ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2003. Т. 34. ВЫП. 7

УДК 530.145.1 + 539.1.01

NONCOMMUTATIVE QUANTUM MECHANICS IN THE PRESENCE OF MAGNETIC FIELD*

S. Bellucci[1], *A. Nersessian*[2,3], *C. Sochichiu*[1,4]

[1]INFN, Laboratori Nazionali di Frascati, Italy
[2]Yerevan State University, Armenia
[3]Yerevan Physics Institute, Armenia
[4]Joint Institute for Nuclear Research, Dubna

*Talk presented by A. N. at the IX International Conference on Symmetry Methods in Physics, Yerevan, Armenia, July 3–8, 2001.

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА
2003. Т. 34. ВЫП. 7

УДК 530.145.1 + 539.1.01

NONCOMMUTATIVE QUANTUM MECHANICS IN THE PRESENCE OF MAGNETIC FIELD*

S. Bellucci[1], *A. Nersessian*[2,3], *C. Sochichiu*[1,4]

[1]INFN, Laboratori Nazionali di Frascati, Italy
[2]Yerevan State University, Armenia
[3]Yerevan Physics Institute, Armenia
[4]Joint Institute for Nuclear Research, Dubna

Recently it was found that quantum mechanics on noncommutative plane possesses, in the presence of constant magnetic field, a «critical point», where the system becomes effectively one-dimensional, and two different «phases» with qualitatively different properties, which the phases of the planar system originate from, specified by the sign of the parameter $\kappa = 1 - B\theta$. Later on, this observation was generalized for the quantum mechanics on the sphere and hyperboloid. Here we review these results and present some new observations on subject.

Недавно было обнаружено, что квантовая механика на некоммутативной плоскости имеет, в представлении постоянного магнитного поля, «критическую точку», в которой система становится эффективно одномерной, а также существуют две различные «фазы», происходящие из плоской системы, с качественно различными свойствами, определяемыми знаком параметра $\kappa = 1 - B\theta$. Позже этот факт был обобщен для квантовой механики на сфере и гиперболоиде. В нашей работе представлен обзор этих результатов и некоторые новые соображения по данной теме.

INTRODUCTION

Noncommutative quantum field theories have been studied intensively during the last several years owing to their relationship with M theory compactifications [1], string theory in nontrivial backgrounds [2], and quantum Hall effect [3] (see, e. g., [4] for a recent review). At low energies, the one-particle sectors become relevant, which prompted an interest in the study of noncommutative quantum mechanics (NCQM) [5–22] (for some earlier studies of NCQM see [23–25]). In these studies some attention was paid to two-dimensional NCQM in the presence of a constant magnetic field: such systems were considered on a plane [10, 13], torus [11], sphere [10], pseudosphere (Lobachevsky plane, or AdS_2) [19, 22].

---

*Talk presented by A. N. at the IX International Conference on Symmetry Methods in Physics, Yerevan, Armenia, July 3–8, 2001.

NCQM on a plane has a critical point, specified by the zero value of the dimensionless parameter

$$\kappa = 1 - B\theta, \tag{1}$$

where the system becomes effectively one-dimensional [10,13]. Out of the critical point, the rotational properties of the model become qualitatively dependent on the sign of κ: for $\kappa > 0$ the system could have an infinite number of states with a given value of the angular momentum, while for $\kappa < 0$ the number of such states is finite [13] (see also [14]). The NCQM on a (pseudo)sphere originates, in some sense, the «phases» of planar NCQM [15]. An interesting point in the different phases is that the «monopole number» corresponding to the constant magnetic field, is defined in the different way. However, in the planar limit the NCQM on (pseudo)sphere results in «nonconventional», or the so-called «exotic» NCQM [12], where the magnetic field is introduced via «minimal», or symplectic coupling.

The «conventional» two-dimensional noncommutative quantum mechanical system with arbitrary central potential in the presence of a constant magnetic field B, suggested by Nair and Polychronakos, is given by the Hamiltonian [10],

$$\mathcal{H}^{\rm plane} = \frac{\mathbf{p}^2}{2} + V(\mathbf{q}^2), \tag{2}$$

and the operators $\mathbf{p}, \mathbf{q}$ which obey the commutation relations

$$[q_1, q_2] = i\theta, \quad [q_\alpha, p_\beta] = i\delta_{\alpha\beta}, \quad [p_1, p_2] = iB, \quad \alpha, \beta = 1, 2, \tag{3}$$

where the noncommutativity parameter $\theta > 0$ has the dimension of *length*.

The difference of the «exotic» NCQM suggested by Duval and Horvathy [12] from the «conventional» planar NCQM lies in the coupling of an external magnetic field. Instead of a naive, or algebraic approach, used in conventional NCQM, the minimal, or symplectic, coupling is used there, in the spirit of Souriau [26]. This coupling assumes that the closed two-form describing the magnetic field is added to the symplectic structure of the underlying Hamiltonian mechanics

$$\big(\mathcal{H}^{\rm plane}, \omega_0 = \theta dp_1 \wedge dp_2 + d\mathbf{q} \wedge d\mathbf{p}\big) \to (\mathcal{H}^{\rm plane}, \omega_0 + B dq_1 \wedge dq_2). \tag{4}$$

The corresponding quantum-mechanical commutators (out of the point $\kappa = 0$) read

$$[q_1, q_2] = i\frac{\theta}{\kappa}, \quad [q_\alpha, p_\beta] = i\frac{\delta_{\alpha\beta}}{\kappa}, \quad [p_1, p_2] = i\frac{B}{\kappa}. \tag{5}$$

The Hamiltonian is the same as in the «conventional» NCQM, (2).

It is convenient to represent these systems as follows:

$$\mathcal{H}^{\text{plane}} = \frac{(\boldsymbol{\pi} + \mathbf{q}/\theta)^2}{2} + V(\mathbf{q}^2), \tag{6}$$

where the operators $\boldsymbol{\pi}$ and $\mathbf{q}$ are given by the expressions

$$\begin{gathered} \pi_1 = p_2 - \frac{q_1}{\theta}, \quad -\pi_2 = p_1 + \frac{q_2}{\theta}, \quad [\pi_\alpha, q_\beta] = 0, \\ \begin{cases} [\pi_1, \pi_2] = -i\kappa/\theta, & [q_1, q_2] = i\theta \quad \text{conventional}, \\ [\pi_1, \pi_2] = -i/\theta, & [q_1, q_2] = i\theta/\kappa \quad \text{exotic}. \end{cases} \end{gathered} \tag{7}$$

The angular momentum of these systems is defined by the operator (out of the point $\kappa = 0$)

$$L = \begin{cases} \mathbf{q}^2/2\theta - \theta\boldsymbol{\pi}^2/2\kappa & \text{conventional}, \\ \kappa\mathbf{q}^2/2\theta - \theta\boldsymbol{\pi}^2/2 & \text{exotic}. \end{cases} \tag{8}$$

Its eigenvalues are given by the expression

$$l = \pm(n_1 - \operatorname{sgn} \kappa\, n_2), \quad n_1, n_2 = 0, 1, \dots, \tag{9}$$

where (n_1, n_2) define, respectively, the eigenvalues of the operators $(\mathbf{q}^2, \boldsymbol{\pi}^2)$ for the «conventional» NCQM and of the $(\boldsymbol{\pi}^2, \mathbf{q}^2)$ for the «exotic» one, the upper sign corresponds to the «conventional» system, and the lower sign to the «exotic» one. Hence, the rotational properties of NCQM qualitatively depend on the sign of κ.

At the «critical point», i. e., for $\kappa = 0$, these systems become effectively one-dimensional [12, 13]

$$[q_1, q_2] = i\theta, \quad \mathcal{H}_0^{\text{plane}} = \begin{cases} \mathbf{q}^2/2\theta^2 + V(\mathbf{q}^2) & \text{conventional}, \\ V(\mathbf{q}^2) & \text{exotic}. \end{cases} \tag{10}$$

Let us remind [12] that for nonconstant B the Jacobi identities failed in the «conventional» model, while in the «exotic» model the Jacobi identities hold for any $B = A_{[1,2]}$, by definition. This reflects the different origin of magnetic fields B appearing in these two models. In the «conventional» model, B appears as the strength of a *noncommutative* magnetic field, while in the «exotic» model, B appears as a *commutative* magnetic field, obtained by the Seiberg–Witten map from the noncommutative one. In the quantum-mechanical context this question was considered in [5].

Notice, that the above «phases» correspond to the diamagnetic and paramagnetic properties of noncommutative electronic gas [14]. While the noncommutativity itself has a straight relation with «conventional» physics. For example, the system under consideration, that is the two-dimensional noncommutative mechanics with constant magnetic field, could be viewed as a nonrelativistic anyone with large spin coupled with electric and constant magnetic fields (compare with [27]).

OSCILLATOR ON NONCOMMUTATIVE PLANE

Let us exemplify the arising of «phases», on the simplest exactly-solvable systems of the mentioned type, that is the harmonic oscillator.

For nonzero κ it is convenient to introduce the operators

$$a^{\pm} = \frac{x^1 \mp \imath x^2}{\sqrt{2\theta}}, \quad b^{\pm} = \frac{\sqrt{\theta}}{\hbar}\frac{\pi_1 \mp \imath\pi_2}{\sqrt{2|\kappa|}}, \tag{11}$$

with the following nonzero commutators

$$[a^-, a^+] = 1, \quad [b^-, b^+] = -\operatorname{sgn}\kappa. \tag{12}$$

In terms of these operators the Hamiltonian (6) is of the form

$$\mathcal{H} = \frac{\hbar^2}{2\mu\theta}\Big(|\kappa|\{b^+b^-\} - 2i\sqrt{|\kappa|}(b^+a^- - a^+b^-) + \{a^+a^-\}\Big) + \\ + V(\theta\{a^+a^-\}). \tag{13}$$

The rotational symmetry of the system corresponds to the conserved angular momentum given by the operator,

$$L = \frac{a^+a^- + a^+a^-}{2} - \operatorname{sgn}\kappa\frac{b^+b^- + b^+b^-}{2}, \quad [\mathcal{H}, L] = 0. \tag{14}$$

Let us introduce the orthonormal basis in the Hilbert space consisting of states

$$|n_a, n_b\rangle = \frac{(a^+)^{n_a}(b^{-\operatorname{sgn}\kappa})^{n_b}}{\sqrt{n_a! n_b!}}\,|0,0\rangle\,, \quad a^-\,|0, n_1\rangle = b^{-\operatorname{sgn}\kappa}\,|n_2, 0\rangle = 0, \tag{15}$$

where $b^{-\operatorname{sgn}\kappa} = b^-$ for $\kappa > 0$, and $b^{-\operatorname{sgn}\kappa} = b^+$ for $\kappa < 0$.

Hence, the angular momentum corresponds to the total occupation number

$$l = n_a + \frac{1}{2} - \operatorname{sgn}\kappa\left(n_b + \frac{1}{2}\right), \quad n_a, n_b = 0, 1, \ldots \tag{16}$$

One can see that the spectrum has different structure depending on the sign of κ: the angular momentum l and the occupation number n_a take the values

$$\begin{array}{lll} n_a = 0, 1, \ldots, & l = n_a, n_a + 1, \ldots & \text{for } \kappa < 0, \\ n_a = 0, 1, \ldots, & l = -\infty, \ldots, -1, 0, \ldots, n_a & \text{for } \kappa > 0. \end{array} \tag{17}$$

Let us consider how these phases appear in the noncommutative circular oscillator, i. e., when

$$V = \frac{\mu\omega^2\mathbf{x}^2}{2}. \tag{18}$$

At the critical point $\kappa = 0$, the system reduces to one-dimensional oscillator with the energy spectrum

$$E^{\rm osc}_{(0)n} = \frac{\hbar^2\mathcal{E}}{\mu\theta}(n+1/2), \quad n = 0, 1, 2, \dots , \tag{19}$$

where we have used the notation

$$\mathcal{E} = 1 + (\mu\omega\theta/\hbar)^2. \tag{20}$$

For $\kappa \neq 0$ let us diagonalize the Hamiltonian, performing the appropriate (pseudo)unitary transformation:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to U \begin{pmatrix} a \\ b \end{pmatrix}, \tag{21}$$

where the matrix U belongs to $SU(1,1)$ for $\kappa > 0$ and to $SU(2)$ for $\kappa < 0$,

$$U = \begin{cases} \begin{pmatrix} \cosh\,\chi\, {\rm e}^{i\phi} & \sinh\,\chi\, {\rm e}^{i\psi} \\ \sinh\,\chi\, {\rm e}^{-i\psi} & \cosh\,\chi\, {\rm e}^{-i\phi} \end{pmatrix} & \text{for } \kappa > 0, \\ \begin{pmatrix} \cos\,\chi\, {\rm e}^{i\phi} & \sin\,\chi\, {\rm e}^{i\psi} \\ -\sin\,\chi\, {\rm e}^{-i\psi} & \cos\,\chi\, {\rm e}^{-i\phi} \end{pmatrix} & \text{for } \kappa < 0. \end{cases} \tag{22}$$

The Hamiltonian becomes diagonal, when ϕ, ψ, χ obey the conditions

$$\begin{gathered} \cos(\phi+\psi) = 0, \\ \begin{cases} (\mathcal{E}+\kappa)\sinh\,2\chi - 2\sqrt{\kappa}\cosh\,2\chi\sin(\phi+\psi) = 0 & \text{for } \kappa > 0, \\ (\mathcal{E}+\kappa)\sin\,2\chi + 2\sqrt{-\kappa}\cos\,2\chi\sin(\phi+\psi) = 0 & \text{for } \kappa < 0. \end{cases} \end{gathered} \tag{23}$$

In that case the Hamiltonian takes the form

$$\mathcal{H}_{\rm osc} = \frac{1}{2}\hbar\omega_-(b^+b^- + b^-b^+) + \frac{1}{2}\hbar\omega_+(a^+a^- + a^-a^+), \tag{24}$$

where

$$\frac{2\mu\theta\omega_\pm}{\hbar} = $$
$$= \begin{cases} \pm(\kappa-\mathcal{E}) + (\mathcal{E}+\kappa)\cosh 2\chi - 2\sqrt{\kappa}\sinh 2\chi\sin(\phi+\psi) & \text{for } \kappa > 0, \\ (\mathcal{E}-\kappa) \pm \left[(\mathcal{E}+\kappa)\cos 2\chi - 2\sqrt{-\kappa}\sin 2\chi\sin(\phi+\psi)\right] & \text{for } \kappa < 0. \end{cases} \tag{25}$$

Then, after some work we get

$$\frac{2\mu\theta\omega_\pm}{\hbar} = \begin{cases} \pm(\mathcal{E}-\kappa) + \sqrt{(\mathcal{E}+\kappa)^2 - 4\kappa} & \text{for } \kappa > 0, \\ (\mathcal{E}-\kappa) \pm \sqrt{(\mathcal{E}+\kappa)^2 - 4\kappa} & \text{for } \kappa < 0. \end{cases} \tag{26}$$

Consequently, the spectrum is of the form

$$\begin{aligned} E^{\rm osc}_{n_a,n_b} &= \hbar\omega_+(n_a + 1/2) + \hbar\omega_-(n_b + 1/2) = \\ &= \frac{\hbar^2}{\mu\theta}\Big[\Big(\sqrt{(\mathcal{E}-\kappa)^2 + 4\kappa(\mathcal{E}-1)}(n_a+1/2)\Big) - \\ &\qquad - \Big(\sqrt{(\mathcal{E}-\kappa)^2 + 4\kappa(\mathcal{E}-1)} + \kappa - \mathcal{E}\Big)\, l\Big]. \end{aligned} \tag{27}$$

Since the transformation (22) belongs to the symmetry group of the rotational momentum L, the magnetic number is given by the same equation as above, (16).

It is seen, that in the $\kappa \to 0$ limit we get the expression (19), with $n = n_a$ and $n_b = 0$. The expressions (17) can be obtained from the requirement of the positivity of the energy spectrum.

Let us remind, that n_a defines the eigenvalue of the operator $|\mathbf{x}|^2/2\theta$, and has a meaning of quantized radius of the system $r_n^2 = \theta(2n_a + 1)$. Hence, at the given point, an increasing/decreasing of the angular momentum l decreases/increases the energy value both for $\kappa > 0$ and for $\kappa < 0$.

We conclude this consideration by the following remarks:

• In the case of the Landau problem, $\mathcal{E} = 1$ (equivalently, $\omega = 0$), one of the frequencies vanishes, and the spectrum reads

$$E_n = \frac{e|B|}{c\mu\hbar}\left(n + \frac{1}{2}\right), \quad l = n_a - \operatorname{sgn}\kappa\, n_b, \quad n = \begin{cases} n_b = 0, 1, \dots & \text{for } \kappa > 0, \\ n_a = 0, 1, \dots & \text{for } \kappa < 0. \end{cases}$$

Hence, though the energy spectrum of the Landau problem does not depend on the noncommutativity parameter, its dependence on the angular momentum essentially depends on $\operatorname{sgn}\kappa$.

• There is an «isotropic point» there, $\mathcal{E} = \kappa > 1$, where the frequencies become equal to each other $\omega_\pm^{\rm isotr} = \omega\sqrt{1 + (\mu\omega\theta/\hbar)^2}$, and the system has a symmetry of an ordinary circular oscillator. In that case the spectrum reads

$$E_n = \omega\sqrt{1 + (\mu\omega\theta/\hbar)^2}(n+1), \quad l = n_a - n_b, \quad n = n_a + n_b.$$

• At the commutative limit, $\theta \to 0$, the effective frequencies read

$$\omega^0_\pm = \pm\frac{eB}{2c\mu\hbar} + \sqrt{\omega^2 + \left(\frac{eB}{2c\mu\hbar}\right)^2}. \tag{28}$$

Hence, we recovered the standard expression for the circular oscillator in a constant magnetic field.

PHASES ON NONCOMMUTATIVE (PSEUDO)SPHERE

The Hamiltonian of the axially-symmetric NCQM on the sphere [10, 18, 20] and pseudosphere [19, 22] in the presence of a constant magnetic field, looks precisely as in the commutative case (up to the dimensionless parameter γ)

$$\mathcal{H} = \pm\gamma\frac{J^2 - s^2}{2r_0^2} + V(\mathbf{x}^2), \tag{29}$$

where the rotation and position operators $J_i = (\mathbf{J}, J_3)$, $x_i = (\mathbf{x}, x_3)$ obey commutation relations

$$\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk}J^k, \quad [J_i, x_j] = i\epsilon_{ijk}x^k, \\ &[x_i, x_j] = i\lambda\epsilon_{ijk}x^k, \quad i, j, k = 1, 2, 3. \end{aligned} \tag{30}$$

Here and after, for squaring the operators and for rising/lowing the indices, we use the diagonal metric diag $(1, 1, 1)$ for the sphere and diag $(-1, -1, 1)$ for the pseudosphere. The upper sign corresponds to a sphere, and the lower one to a pseudosphere. The noncommutativity parameter λ has the dimension of *length* and is assumed to be positive, $\lambda > 0$. The values of the Casimir operators of the algebra are fixed by the equations

$$C_0 \equiv x^2 = r_0^2 > 0, \quad C_1 \equiv Jx - \frac{\lambda J^2}{2} = -r_0 S(s, r_0), \tag{31}$$

where r_0 is the radius of the (pseudo)sphere and s is the «monopole number». In the commutative limit $\lambda \to 0$ the parameters S and γ should have a limit

$$\lambda \to 0 \ \Rightarrow \gamma \to 1, \quad S(s, r_0) \to s = Br_0^2, \tag{32}$$

where B is a strength of the magnetic field.

The angular momentum of the system is defined by the operator J_3: $[\mathcal{H}, J_3] = 0$.

The algebra (30) can be split in two independent copies of $SU(2)/SU(1,1)$,

$$\begin{aligned} K_i = J_i - \frac{x_i}{\lambda} : \quad &[K_i, x_j] = 0, \\ [K_i, K_j] = i\epsilon_{ijk}K^k, \quad &[x_i, x_j] = i\lambda\epsilon_{ijk}x^k. \end{aligned} \tag{33}$$

In these terms the Casimir operators read $C_0 = x^2$ and $C_1 = \lambda(x^2 - K^2)/2$. For the NCQM on a sphere, the Casimir operators C_0, K^2 are positive. For a pseudosphere C_1 is positive, whereas another Casimir operator, i. e., K^2, could get positive, zero or negative values. We restrict ourselves to the case of positive

K^2 which is responsible for the description of the discrete part of the energy spectrum. Hence, the Casimir operators take the following values:

$$r_0^2 = \lambda^2 m(m \pm 1), \quad 2sr_0 + \ldots = \lambda[k(k \pm 1) - m(m \pm 1)], \tag{34}$$

where m, k are non-negative (half)integers fixing the representation of $SU(2)$, in the case of sphere, and $m, k > 1$ are real numbers, fixing the representation of $SU(1,1)$, in the case of pseudosphere.

To obtain the planar limit of the NCQM on the (pseudo)sphere out of the point $\kappa = 0$, we shoud take the limits [10]

$$k \to \infty, \quad m \to \infty, \tag{35}$$

and consider small neighborhoods of the «poles» of «coordinate and momentum spheres»

$$\begin{aligned} x_3 &\approx \epsilon_1 \left(r_0 \mp \frac{\mathbf{x}^2}{2r_0} \right) = \epsilon_1 \lambda \left(\tilde{m} \mp \frac{\mathbf{x}^2}{2\lambda^2 \tilde{m}} \right), \\ k_3 &\approx \epsilon_2 \left(\tilde{k} \mp \frac{\mathbf{K}^2}{2\tilde{k}} \right), \quad \epsilon_1, \epsilon_2 = \pm 1. \end{aligned} \tag{36}$$

In these neighborhoods the commutation relations

$$[x_1, x_2] \approx i\epsilon_1 \lambda^2 \tilde{m}, \quad [K_1, K_2] \approx i\epsilon_2 \tilde{k} \tag{37}$$

hold, while the Hamiltonian looks as follows:

$$\begin{aligned} \mathcal{H} = \pm\gamma \frac{\tilde{k}^2 \pm 2\mathbf{x}\mathbf{K}/\lambda + 2k_3 x_3/\lambda + \tilde{m}^2 - s^2}{2r_0^2} + V(\mathbf{x}^2) \approx \\ \approx \mathcal{E}_0 - \epsilon\gamma \frac{(\nu\mathbf{K} - \epsilon\mathbf{x}/\lambda\nu)^2}{2r_0^2} + V(\mathbf{x}^2). \end{aligned} \tag{38}$$

Here we introduced the notation

$$\tilde{m} = \sqrt{m(m \pm 1)}, \quad \tilde{k} = \sqrt{k(k \pm 1)}, \quad \nu = \sqrt{\tilde{m}/\tilde{k}}, \quad \epsilon = \epsilon_1 \epsilon_2,$$

and

$$\mathcal{E}_0 = \pm\gamma \frac{(\tilde{k} + \epsilon\tilde{m})^2 - s^2}{2r_0}. \tag{39}$$

In order to get the planar Hamiltonian with a positively defined kinetic term, we should put

$$\operatorname{sgn} \gamma = -\epsilon. \tag{40}$$

For a correspondence with the planar Hamiltonian (6), we redefine the coordinates and momenta of the resulting system as follows:

$$\boldsymbol{\pi} = \frac{\sqrt{|\gamma|}\nu \mathbf{K}}{r_0}, \quad \frac{\mathbf{q}}{\theta} = \frac{\sqrt{|\gamma|}\mathbf{x}}{\nu\lambda r_0}. \tag{41}$$

Then, comparing their commutators with (7), we get the following expressions for the θ parameters:

$$\theta = \begin{cases} \lambda^2 \tilde{m}^2/\gamma\tilde{k} & \text{conventional,} \\ \lambda^2\tilde{m}/\gamma & \text{exotic,} \end{cases} \tag{42}$$

and the same value of κ for both systems

$$\kappa = -\epsilon\frac{\tilde{m}}{\tilde{k}}. \tag{43}$$

Naively, it seems that the planar NCQM with $\kappa < 0$ and positive kinetic term corresponds to a (pseudo)spherical system with negative kinetic term. Fortunately, thanks to the additional term $-\gamma s^2/2r_0^2$ the kinetic term of the Hamiltonian (29) remains positively defined! Indeed, one can identify the monopole number s as follows:

$$s = \begin{cases} \tilde{m} + \epsilon\tilde{k} & \text{conventional,} \\ -(\tilde{m} + \epsilon\tilde{k}) & \text{exotic,} \end{cases} \tag{44}$$

which yields the vanishing of the «vacuum energy» (39), and the following expressions for the magnetic field, which are in agreement with (32):

$$\tilde{B} = \frac{B}{1 - B\theta} = \frac{1-\kappa}{\theta\kappa} = \begin{cases} \gamma s/\kappa r_0^2 & \text{conventional,} \\ \gamma s/r_0^2 & \text{exotic.} \end{cases} \tag{45}$$

One can redefine the parameters s, κ as follows:

$$\kappa = -\epsilon\frac{m \pm 1/2}{k \pm 1/2}, \quad s = \begin{cases} k \pm 1/2 + \epsilon(m \pm 1/2) & \text{conventional,} \\ -(k \pm 1/2) - \epsilon(m \pm 1/2) & \text{exotic.} \end{cases} \tag{46}$$

In this case the monopole number is quantized on a sphere, and it remains not quantized on a pseudosphere, as in the commutative case. The constant energy term $\mathcal{E}_0$ vanishes upon this choice, too.

Taking into account that the maximal value of J^2 is $(k+m)(k+m\pm 1)$, and the minimal one is $|k-m|(|k-m| \pm 1)$, we obtain

$$\pm\epsilon\frac{J^2 - s^2}{2r_0^2} \geq 0. \tag{47}$$

Hence, the kinetic part of the (pseudo)spherical Hamiltonian is positively defined for any γ. Expanding (pseudo)spherical NCQM near the upper/lower bound of J^2, we shall get the planar NCQM with $\kappa > 0/\kappa < 0$.

In order to avoid the rescaling of the potential in the planar limit, we should take

$$\gamma = \begin{cases} \kappa \Rightarrow \lambda = \theta/r_0 & \text{conventional,} \\ 1/\kappa \Rightarrow \lambda = \theta/\kappa r_0 & \text{exotic.} \end{cases} \tag{48}$$

Upon this choice, the expression (45) reads

$$\frac{s}{r_0^2} = \begin{cases} \tilde{B} & \text{conventional,} \\ B & \text{exotic.} \end{cases} \tag{49}$$

In the «conventional» picture $\tilde{B}$ plays the role of the strength of a (commutative) magnetic field obtained by the Seiberg–Witten map from the noncommutative one [10]. In the «exotic» picture the same role is played by B. Hence, in both pictures we get the standard expression for the strength of the constant commutative magnetic field on the (pseudo)sphere, and the quantization of the flux of the commutative magnetic field on the sphere, as well.

We did not consider yet the planar limit of the critical point of the (pseudo)spherical NCQM, and did not establish yet, whether the latter results in the «conventional» or in the «exotic» planar NCQM, in this limit. For this purpose let us notice, that our specification of the «monopole number» s and of the γ parameter yields the following values of the first Casimir operator:

$$C_0 = r_0^2 = \lambda^2 \tilde{m}^2 \;\Rightarrow r_0^2 = \begin{cases} \theta\tilde{m} & \text{conventional,} \\ \theta\tilde{k} & \text{exotic.} \end{cases} \tag{50}$$

Thus, in the «conventional» picture, the (pseudo)spherical NCQM becomes one-dimensional for $\tilde{k} = 0$, i. e., for $\kappa \to \infty$; in the «exotic» picture we have, instead, $\tilde{m} = 0$, i. e., $\kappa = 0$.

In the «exotic» picture the (pseudo)spherical NCQM in the $\kappa \to 0$ limit results in the system

$$\mathcal{H}_0 = V(\mathbf{x}^2), \quad [x_1, x_2] = i\theta\sqrt{1 \pm \mathbf{x}^2/r_0^2}, \tag{51}$$

which reduces, immediately, to the «exotic» planar NCQM at the critical point.

Hence, the «critical point» and «phases» of (pseudo)spherical NCQM reduce, in the planar limit, to the respective «critical point» and «phases» of «exotic» NCQM, with the symplectic coupling of the (commutative) magnetic field.

The eigenvalues of the angular momentum of the (pseudo)spherical NCQM are given by the expression

$$j_3 = k_3 + m_3 \times$$
$$\times \begin{cases} k_3 = 0, \pm 1, \dots, \pm k, & m_3 = 0, \pm 1/2, \dots, \pm m \quad \text{sphere}, \\ k_3 = \pm k, \pm(k+1), \dots, & m_3 = \pm m, \pm(m+1), \dots \quad \text{pseudosphere}. \end{cases} \tag{52}$$

Introducing $m_3 = \epsilon_1(m \mp n_1)$, $k_3 = \epsilon_1(k \mp n_2)$, we get

$$j_3 = \epsilon_1(m \mp n_1) + \epsilon_2(k \mp n_2) = \epsilon_1((m + \epsilon k) \mp (n_1 + \epsilon m_2)), \tag{53}$$

which is in agreement with the angular momentum of the planar NCQM (9).

As is known, there is a well-known Levi–Civita–Bohlin transformation $w = z^2$ which connects planar/pseudospherical circular oscillator with two-dimensional planar/pseudospherical Coulomb problem [28]. It seems to be attractive, performing similar transformation to noncommutative oscillator, to obtain the exactly-solvable noncommutative two-dimensional Coulomb system. Unfortunately, it is easy to see, that the resulting system could not be reduced to the two-dimensional Coulomb system with a constant noncommutativity parameter.

Acknowledgments. This work was supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime, the Iniziativa Specifica MI12 of the Commissione IV of INFN (S. B.), the INTAS grants 00-0254 (S. B) a 00-00262 (A. N. and C. S.), and the ANSEF grant PS81 (A. N.).

REFERENCES

1. *Connes A., Douglas M. R., Schwarz A.* // JHEP. 1998. V. 9802. P. 003.
2. *Seiberg N., Witten E.* // JHEP. 1999. V. 9909. P. 032.
3. *Susskind L.* hep-th/0101029;
Polychronakos A. P. // JHEP. 2001. V. 0104. P. 011;
Nair V. P., Polychronakos A. P. // Phys. Rev. Lett. 2001. V. 87. P. 030403.
4. *Szabo R. J.* hep-th/0109162.
5. *Mezincescu L.* hep-th/0007046.
6. *Zachos C.* // Intern. J. Mod. Phys. A. 2002. V. 17. P. 297.
7. *Nair V. P.* // Phys. Lett. B. 2001. V. 505. P. 249.
8. *Chaichian M., Sheikh-Jabbari M. M., Tureanu A.* // Phys. Rev. Lett. 2001. V. 86. P. 2716.
9. *Gamboa J., Loewe M., Rojas J. C.* // Phys. Rev. D. 2001. V. 64. P. 067901.
10. *Nair V. P., Polychronakos A. P.* // Phys. Lett. B. 2001. V. 505. P. 267.
11. *Morariu B., Polychronakos A. P.* // Nucl. Phys. B. 2001. V. 610. P. 531.

12. *Duval C., Horvathy P. A.* // J. Phys. A. 2001. V. 34. P. 10097;
Horvathy P. A. hep-th/0201007.
13. *Bellucci S., Nersessian A., Sochichiu C.* // Phys. Lett. B. 2001. V. 522. P. 345.
14. *Dayi O. F., Jellal A.* // Phys. Lett. A. 2001. V. 287. P. 349.
15. *Bellucci S., Nersessian A.* // Phys. Lett. B. 2002. V. 542. P. 295.
16. *Acatrinei C.* // JHEP. 2001. V. 0109. P. 007.
17. *Correa D. H. et al.* // Ibid. V. 0111. P. 034.
18. *Karabali D., Nair V. P., Polychronakos A. P.* // Nucl. Phys. B. 2002. V. 627. P. 565.
19. *Iengo R., Ramachandran R.* // JHEP. 2002. V. 0202. P. 017.
20. *Deriglazov A. A.* hep-th/0112053; Phys. Lett. B. 2002. V. 530. P. 235.
21. *Ho P. M., Kao H. C.* // Phys. Rev. Lett. 2002. V. 88. P. 151602.
22. *Morariu B., Polychronakos A. P.* hep-th/0201070.
23. *Dunne G. V., Jackiw R., Trugenberger C. A.* // Phys. Rev. D. 1990. V. 41. P. 661.
24. *Madore J.* // Phys. Lett. B. 1991. V. 263. P. 245.
25. *Lukierski J., Stichel P. C., Zakrzewski W. J.* // Ann. Phys. 1997. V. 260. P. 224.
26. *Souriau J.-M.* Structure des systemes dinamiques. Paris: Dunod, 1970.
27. *Chou C. H., Nair V. P., Polychronakos A. P.* // Phys. Lett. B. 1993. V. 304. P. 105.
28. *Nersessian A., Ter-Antonyan V., Tsulaia M. M.* // Mod. Phys. Lett. A. 1996. V. 11. P. 1605;
Nersessian A., Pogosyan G. // Phys. Rev. A. 2001. V. 63. P. 020103(R).