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CONSTRUCTION OF EXTREMAL VARIATIONAL PRINCIPLES IN NON-EULER FUNCTIONAL CLASSES FOR PDEs OF SECOND ORDER WITH NONPOTENTIAL OPERATORS

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Given a linear PDE of second order

$$\left.\begin{aligned} L[u] \equiv \sum_{i,j=1}^{n} a_{ij} \tfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \tfrac{\partial u}{\partial x_i} + \lambda u = g(x) \\ x = (x_1, ..., x_n); u(x) \in D(L) = C^2(\Omega) \cap C^1(\overline{\Omega}) \cap C \circ (\Omega) \end{aligned}\right\} \quad (1)$$

and a class of Euler–Lagrange functionals of a sort

$$\Phi[u] = \int_\Omega F(x, u(x), u_1', ..., u_n')dx, u_i' = \frac{\partial u}{\partial x_i}, \quad (2)$$

one considers a problem of finding functionals $\Phi[u]$ (2) and variational factors — sufficiently smooth functions $M = M(x, u(x), u_1', ..., u_n')$ such that $M \neq 0$ in Ω and

$$\delta\Phi[y] = \int_\Omega M\{L[u] - g\}\delta u dx \, \forall u \in D(L). \quad (3)$$

Evidently that from above mentioned and grace to conditions on M and $D(L)$ the variational principle follows:

$$\delta\Phi[u] = 0 \Leftrightarrow L[u_0] = g(x). \quad (4)$$

Using the approach of paper [1], the next results are established.

1. If equation (1) is of parabolic or ultra parabolic type in Ω, there don't exist such variational factors $M = M(x, u(x), u_1', ..., u_n')$ and functional $\Phi[u]$ that relation (3) would be fulfilled.

This result generalizes numerous special cases (look at [2]), when one considered only squared with respect to u, $u_1', ..., u_n'$ functionals $\Phi[u]$ (2) and factors M of a sort $M = M(x)$.

2. If equation (1) is of hyperbolic type in Ω there do not exist factors $M = M(x, u(x), u'_1, ..., u'_n)$ and don't exist bounded below functionals $\Phi[u]$ (2) such that relation (3) would be fulfilled.

3. Let (1) be an equation of elliptic Ω, so that $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \mu|\xi|^2$, $\mu > 0$. Then there exists $\lambda_0 = \lambda_0(\Omega, b_1, ..., b_n)$ such that for $\forall\lambda < \lambda_0$ there do not exist bounded below on $D(L)$ functionals $\Phi[u]$ (2) and $M = M(x, u(x), u'_1, ..., u'_n)$ such that relation (3) would be fulfilled.

Thus obtained results exhibit evident unsufficiency of a class of Euler–Lagrange functionals of a sort (2) in order to construct extremal variational principles even in generalized sense (3), (4) for all three basic types of PDE. From this follows that in order to get such extremal variational principles it's necessary either to find functional in other, different from generally accepted classes of Euler–Lagrange functionals (2) or to substitute a variational factor M in (3) by an auxiliary linear invertible on $R(B)$, $\overline{R(B)} = L_2(\Omega)$ operator $B, D(B) = D(L)$ such that instead of (3) would be fulfilled a relation

$$\delta\Phi[u] = \int_{\Omega} B \cdot \{L[u] - g\}\delta u dx = 0\, \forall u \in D(L). \tag{3a}$$

It's evident that from (3a) grace to conditions on $B, D(B) = D(L)$ follow also relations (4).

Such approach allowing to construct extremal variational principles for PDE, including among them ones of hyperbolic and parabolic types, were developed in [3] (look also review in [2]).

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