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NEW RESULTS IN THE THEORY OF ANALYTIC FUNCTIONALS WITH APPLICATIONS TO GAUGE QFT

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A general scheme for the operator realization of infinite series of Wick powers of the indefinite metric free field is developed. It is based on new results in the theory of analytic functionals which reduce the problem of finding the functional domain of definition of the Wick series to an elementary estimation of the infrared and ultraviolet behavior of the Hilbert majorant form associated with the indefinite metric and enables us to prove that their sums satisfy all requirements of the pseudo-Wightman formalism in the most general situation when the Poincaré group is implemented by unbounded pseudounitary operators.

We intend to outline some recent results concerning properties of analytic functionals, i.e., of the linear continuous functionals defined on analytic test functions, which may be useful for the nonperturbative study of gauge models. Their efficiency will be demonstrated by rigorous constructing infinite series of Wick powers of the free fields whose correlation functions exhibit infrared singularities violating the positivity condition. Objects of this kind enter as building blocks into the exact solutions of some simple gauge models and a more deep insight into their operator realization may be instructive for constructing a consistent Euclidean version of indefinite metric QFT. The choice of adequate test function space and the formulation of the spectral condition for infrared singular fields turn out to be nontrivial problems which were first raised in [1,2]. The presented results develop the approach proposed in [3] and, as it seems to us, give a complete answer to such questions. They reveal those properties of ultradistributions and hyperfunctions that hold for wider classes of generalized functions. We show that the structural theorems of distribution theory exploited in the Wightman axiomatic approach have analogues for functionals of arbitrarily severe singularity. We consider arbitrary Wick-ordered power series and make use of generalized Gelfand–Shilov's test function spaces of type S. Let us recall that the space S^b_a in this class is determined by two nondecreasing sequences a_k and b_l of positive numbers and consists of those smooth functions on $\mathbf{R}^n$ that satisfy the bounds

$$\sup_x \sup_{|\kappa|\le k} \sup_{|\lambda|\le l} |x^k \partial^l f(x)| \le C A^k B^l a_k b_l,$$

with constants A, B, C depending on f. From the standpoint of functional analysis it is reasonable to impose the conditions

$$a_k^2 \leq a_{k-1}a_{k+1}, \quad a_{k+l} \leq C_1 h_1^{k+l} a_k a_l, \quad b_k^2 \leq b_{k-1}b_{k+1}, \quad b_{k+l} \leq C_2 h_2^{k+l} b_k b_l,$$

where $C_{1,2}$ and $h_{1,2}$ are constants. The indicator functions $a(r) = \sup_k r^k/a_k$ and $b(s) = \sup_k s^k/b_k$ characterize the behavior of the test functions and of their Fourier transforms at infinity and, in the context of QFT, they indicate the infrared and ultraviolet behavior of the fields defined on S_a^b. We consider local field theory and assume that there is no lack of test functions of compact support in configuration space, which is equivalent to the condition

$$\sum_k b_k^{-1/k} < \infty. \tag{1}$$

On the contrary, in momentum space the operator realization of gauge models in a generic covariant gauge is possible, as a rule, only on a space consisting of analytic test functions, see [1,2]. So we do not impose such a restriction on a_k. In order to construct a Wick power series we first need to find the test function space on which the series is convergent. In the case of an indefinite metric, for this purpose we make use of the following theorem proven in [4].

Theorem 1. *Let V be an open convex cone and let (v_K) be a countable family of tempered distributions which are the boundary values of functions $\mathbf{v}_K(z)$ holomorphic in the tubular domain $T^V = \{z : \operatorname{Im} z \in V\}$. If there exists a vector $\eta \in V$ such that*

$$\sum_K \inf_{0<t<\delta} e^{st} \int \frac{|\mathbf{v}_K(x + it\eta)|}{a(|x|/A)} \, \mathrm{d}x \leq C_{\delta,\epsilon,A} b(\epsilon s)$$

for every positive δ, ϵ and A, then the family (v_K) is unconditionally summable in the space ${S'}_a^b$ dual to S_a^b.

Let $\phi(x)$ be a neutral scalar free field acting in a pseudo-Hilbert space $\mathcal{H}$ of states. In order to show how Theorem 1 can be applied for constructing the Wick series

$$\sum_k d_k : \phi^k : (x), \tag{2}$$

we recall [5] that $\mathcal{H}$ is provided with an auxiliary positive scalar product $(\cdot, \cdot)$ related to the indefinite metric $\langle \cdot, \cdot \rangle$ by the relation $\langle \Phi, \Psi \rangle = (\Phi, \theta\Psi) \quad (\Phi, \Psi \in \mathcal{H})$, where θ is a self-adjoint involutory operator called Krein operator. Convergence of the series (2) should be established in the topology defined by the form $(\cdot, \cdot)$. This form is consistent with the Fock structure of $\mathcal{H}$ and is completely determined by presetting at the one-particle level, where it appears as a positive majorant of the two-point correlation function of the field $\phi(x)$. In particular, the following formulas are valid:

$$\langle \phi(f)\Psi_0, \phi(g)\Psi_0\rangle = \int w(x-x')\bar{f}(x)g(x')\,\mathrm{d}x\mathrm{d}x',$$

$$(\phi(f)\Psi_0, \phi(g)\Psi_0) = \int w_{\mathrm{maj}}(x,x')\bar{f}(x)g(x')\,\mathrm{d}x\mathrm{d}x',$$

where Ψ_0 is the vacuum state and f, g are test functions. The majorant is, in general, a more singular distribution than w itself. That is why the operator realization of Wick series is possible only under some additional restrictions on test functions as compared to those ensuring the existence of the Wightman functions of their sums. For example, the series (2) smeared with a test function f converges on the vacuum vector if and only if the number series

$$\sum_k \|d_k : \phi^k : (f)\Psi_0\|^2 = \sum_k k!|d_k|^2 \int w_{\mathrm{maj}}(x,x')^k \bar{f}(x) f(x')\mathrm{d}x\,\mathrm{d}x'$$

is convergent, while the two-point function of the field φ defined by (2) is of the form

$$\sum_k k! d_k^2 \int w(x-x')^k f(x)g(x')\mathrm{d}x\,\mathrm{d}x'. \tag{3}$$

The locality condition imposes on the coefficients of the series the restriction

$$\lim_{k\to\infty} |d_k|^{1/k} = 0 \tag{4}$$

which ensures the analyticity of the vacuum expectation value (3) in the usual domain of local theory because (3) is the composition of an entire function and the Wightman function w. The majorant should not necessarily be translation invariant, but by its sense it inherits the spectral properties of w, i.e., $\operatorname{supp} \tilde{w}_{\mathrm{maj}}(p,p') \subset \bar{V}_+ \times \bar{V}_-$. This implies that in configuration space the majorant is the boundary value of a function $\mathbf{w}_{\mathrm{maj}}$ holomorphic in the tube $\{z, z' : \operatorname{Im} z \in V_-, \operatorname{Im} z' \in V_+\}$. From Theorem 1, it follows that the series (2) smeared with test functions in S^b_a converges on the vacuum vector if there exists $\eta \in V_+$ such that

$$\inf_{0<t<\delta} e^{st} \sup_{x,x'} \sum_k k! d_k \, |w_{\mathrm{maj}}(x - it\eta, x' + it\eta)|^k / a(\epsilon(|x| + |x'|)) \le C_{\delta,\epsilon} b(\epsilon s)$$

for every $\delta, \epsilon > 0$. In order to obtain the complete operator realization of φ with a dense and invariant domain in $\mathcal{H}$, we need to examine convergence of the multiple series

$$\sum_{\{k_j\}} d^{(1)}_{k_1} \dots d^{(n)}_{k_n} : \phi^{k_1} : (f_1) \dots : \phi^{k_n} : (f_n)\Psi_0,$$

where the coefficients $d_k^{(j)}$ are subordinated to those of the series under study in the sense that $|d_k^{(j)}| \leq C|d_k|$. This vector series is unconditionally convergent in the Hilbert norm if the number series

$$\sum_{\substack{k_1,\dots,k_n \\ l_1,\dots,l_n}} \Big(\prod_{1\leq j\leq n} d_{k_j} :\phi^{k_j}:(f_j)\Psi_0, \prod_{1\leq j\leq n} d_{l_j} :\phi^{l_j}:(f_j)\Psi_0\Big) = \sum_K D_K\, W^K(\bar{f}\otimes f) \tag{5}$$

is absolutely convergent. Here $f = f_1 \otimes \ldots \otimes f_n$, K is a multi-index with the components $k_{j,m}$, $1 \leq j < m \leq 2n$, and the following designations are used:

$$W^K = \prod_{1\leq j<m\leq n} w(x_m - x_j)^{k_{jm}} \prod_{n+1\leq j<m\leq 2n} w(x_j - x_m)^{k_{jm}} \times$$

$$\times \prod_{\substack{1\leq j\leq n \\ n+1\leq m\leq 2n}} w_{\mathrm{maj}}(x_j, x_m)^{k_{jm}},$$

$$D_K = \frac{\kappa!}{K!} \prod_{1\leq j\leq n} \bar{d}_{\kappa_j} \prod_{n\leq j\leq 2n} d_{\kappa_j}, \quad \kappa_j = k_{1,j} + \ldots + k_{j-1,j} + k_{j,j+1} + \ldots + k_{j,2n}.$$

Because of consistency of the Krein operator with the Fock structure, the series (5) has the same form as the vacuum expectation values of φ, where a part of functions w, however, should be substituted by majorants. If the coefficients d_k obey the condition

$$d_k d_l \leq C h^{k+l} d_{k+l}, \tag{6}$$

then the restrictions on test functions that guarantee the existence of the complete operator realization are practically the same as those ensuring convergence of the series (2) on the vacuum state. In order to get a convenient criterion, let us characterize the infrared and ultraviolet behavior of $\mathbf{w}_{\mathrm{maj}}$ by a pair of monotone nonnegative functions w_{IR}, w_{UV} increasing as their arguments tend to infinity and to zero, respectively, so that

$$|\mathbf{w}_{\mathrm{maj}}(z, z')| \leq C_0 + C_1\, w_{IR}(|z| + |z'|) + C_2\, w_{UV}(|y| + |y'|), \tag{7}$$

with y being in the negative part of the y_0-axis and y' lying in its positive part. Applying Theorem 1, we get the following result.

Theorem 2. *Let $\phi(x)$ be a free field acting in a pseudo-Hilbert space $\mathcal{H}$ and let the positive majorant of its correlation function satisfy the estimate (7) with monotone w_{IR}, w_{UV}. If moreover the conditions (4) and (6) are met, then the Wick-ordered* *power series* (2) *and every series subordinated to it are well-defined as operator-valued generalized functions on every space* S_a^b *whose indicator functions satisfy the inequalities*

$$\sum_k L^k k! d_{2k} w_{IR}(r)^k \le C_{L,\epsilon}\, a(\epsilon r), \quad \inf_{0<t<1} e^{s\tau} \sum_k L^k k! d_{2k} w_{UV}(t)^k \le C_{L,\epsilon}\, b(\epsilon s)$$

for arbitrarily large $L > 0$ *and arbitrarily small* $\epsilon > 0$.

Under the conditions stated above, the linear span of all vectors of the form $\prod_{1\le j\le n} \varphi^{(j)}(f_j)\Psi_0$, with $f_j \in S_a^b$, $n = 1, 2, \dots$, can be taken as an invariant domain $D(\varphi) \subset \mathcal{H}$. We have constructed the operator realization of the series in the space of states of the initial field ϕ and therefore a part of the requirements of pseudo-Wightman formalism [5] are obviously satisfied for the fields defined by them. In particular, they are local, and moreover, they are mutually local. Let the Poincaré group be implemented by pseudounitary operators $U(\xi, \Lambda)$. These operators are closable and, taking into account the transformation law of $\phi(x)$, we see that they can be uniquely extended to $D(\varphi)$ and

$$U(\xi, \Lambda)\varphi(f)\, U(\xi, \Lambda)^{-1}\Psi = \varphi(f_{(\xi,\Lambda)})\, \Psi$$

for every $\Psi \in D(\varphi)$. Because of the density of $S_a^b(\mathbf{R}^d)^{\otimes n}$ in $S_a^b(\mathbf{R}^{nd})$, the multilinear vector-valued functional $\varphi(f_1)\dots\varphi(f_n)\Psi_0$ uniquely defines the vector-valued generalized function

$$\Psi(f) = \int \varphi(x_1)\dots\varphi(x_n)\, f(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n \Psi_0, \qquad f \in S_a^b(\mathbf{R}^{nd}). \tag{8}$$

If the sequence a_k satisfies a condition analogous to (1), then the spectral condition

$$\mathrm{supp} \int \langle \Phi,\, U(\xi, I)\Psi\rangle\, e^{-ip\xi}\, \mathrm{d}\xi \subset \bar{V}_+ \tag{9}$$

holds not only on the initial domain D_0 of the field ϕ, but also for all $\Phi, \Psi \in D(\varphi)$. In this case, however, the integrand is not necessarily polynomially bounded and may increase like $a(\epsilon|\xi|)$. The Fourier transform of (8) is defined on the space $S_a^b = \tilde{S}_b^a$, and by using (9), we find in the ordinary way that its support is contained in the cone

$$\{p \in \mathbf{R}^{nd} : p_m + \dots + p_n \in \bar{V}_- \quad \forall\, m = 1, \dots, n\}. \tag{10}$$

This support property makes no sense, however, if the operator realization is possible only on test functions analytic in momentum space. Its natural generalization was proposed in [6]. For formulating this generalized spectral condition we need a redefinition of the momentum-spaces test functions S^b_a. Namely, we will use spaces similar to those of type W introduced by B.L.Gurevich.

Let γ and β be nonnegative monotone functions on semi-axis, γ being convex and growing no worse than linearly, while β increasing faster than logarithmically. Let us denote by $\mathcal{E}^\gamma_\beta$ the space of all entire functions on ^{d} satisfying the bound

$$|g(p+iq)| \leq C\exp\{\gamma(A|q|) - \beta(|p|/B)\}$$

with constants depending on f. It is easily verified that if the function $\beta(s)$ is convex in respect to $\ln s$ and satisfies the restriction $2\beta(s) \leq C+\beta(hs)$ with some constants C, h, then $\mathcal{E}^\gamma_\beta$ is identified with S^a_b, where $a_k = k!/\sup_{r>0} r^k e^{-\gamma(r)}$ and $b_k = \sup_{s>0} s^k e^{-\beta(s)}$. This subclass of the spaces of type S will be referred to as the Gelfand–Shilov–Gurevich spaces. The condition (4) implies that $\beta(s)$ has less than linear growth and we assume it to be concave. The crucial point is that the analytic functionals belonging to the dual space $\mathcal{E}'^\gamma_\beta$ have an angular localizability property which substitutes for the notion of support. More specifically, for every open cone $U \subset \mathbf{R}^d$, we define $\mathcal{E}^\gamma_\beta(U)$ to be the space of all entire functions satisfying

$$|g(p+iq)| \leq C\exp\{\gamma(A|q|) + \gamma\circ\delta_U(Ap) - \beta(|p|/B)\},$$

where $\delta_U(p)$ is the distance from the point p to the cone U. A closed cone $K \subset \mathbf{R}^n$ is said to be a *carrier cone* of $u \in \mathcal{E}'^\gamma_\beta$, if u can be continuously extended to every space $\mathcal{E}^\gamma_\beta(U)$, where $U \supset K\setminus\{0\}$, or in other words, if u has a continuous extension to the union

$$\mathcal{E}^\gamma_\beta(K) = \bigcup_{U\supset K\setminus\{0\}} \mathcal{E}^\gamma_\beta(U)$$

endowed with the inductive limit topology. Under the assumption that $\mathcal{E}^\gamma_\gamma$ is non-trivial, which is sufficient for the applications of interest, the following theorems have been proved.

Theorem 3. *The space $\mathcal{E}^\gamma_\beta$ is dense in every space $\mathcal{E}^\gamma_\beta(K)$, where K is a closed cone.*

Theorem 4. *Every functional $u \in \mathcal{E}'^\gamma_\beta(K_1\cup K_2)$ allows a decomposition of the form $u = u_1 + u_2$, where $u_j \in \mathcal{E}'^\gamma_\beta(K_j)$, $\quad j = 1, 2$.*

Theorem 5. *If K_1, K_2 are carrier cones of $u \in \mathcal{E}'^\gamma_\beta$, then so is the cone $K_1\cap K_2$.*

As a consequence, there is a unique minimal closed cone K such that $u \in \mathcal{E}'^\gamma_\beta(K)$. It may be called the *quasisupport* of u. These results can be derived in the same manner as for the particular case considered in [7], where $\gamma(s) = s^\lambda$, $\beta(s) = s^\mu$ and $\lambda > 1$, $0 < \mu < 1$.

Let us denote $\alpha(r) = \sup_{s>0}(rs - \gamma(s))$ and $\vartheta(t) = -\inf_{s>0}(ts - \beta(s))$. These functions are nonnegative convex and monotone increasing as $s \to \infty$ and

$t \to 0$, respectively. It is easy to see that every $u \in \mathcal{E}'^{\gamma}_{\beta}$ carried by a properly convex cone K has the Laplace transform $\mathbf{v}(z) = (u, e^{i(z,\cdot)})$ holomorphic in the tubular domain whose base is the interior of the dual cone $K^*{=}\{y : py{\geq}0, \forall p \in K\}$. Moreover $\mathbf{v}(z)$ satisfies the estimate

$$|\mathbf{v}(x+iy)| \leq C_\epsilon(V') \exp\{\alpha(\epsilon|x|) + \vartheta(|y|/\epsilon)\} \quad (y \in V') \tag{11}$$

for every $\epsilon > 0$ and for each cone V' such that $\bar{V}' \setminus \{0\} \subset \operatorname{int} K^*$.

Now, let V be an open connected cone and let $\mathcal{A}^{\vartheta}_{\alpha}(V)$ be a space of functions holomorphic in $\{z : \operatorname{Im} z \in V\}$ and, for any subcone V' of V, satisfying (11), where α is nonnegative convex and monotone increasing faster than linearly, while $\vartheta(t)$ indefinitely increases with decreasing t but is not presupposed to be convex. Under the condition $2\vartheta(t) \leq C + \vartheta(t/h)$, this space is an algebra with respect to the pointwise multiplication. Theorems 3–5 enable us to obtain the following result.

Theorem 6. *The algebra $\mathcal{A}^{\vartheta}_{\alpha}(V)$ is exactly the Laplace transformed space $\mathcal{E}^{\alpha_*}_{(-\vartheta)^*}(V^*)$, where asterisks signify the dual cone of V and the convex and concave functions that are conjugate of α and $-\vartheta$.*

From this it follows that if the hypotheses of Theorem 2 are fulfilled for a space S^b_a whose Fourier transform is in the Gelfand–Shilov–Gurevich class, then the cone (10) is a carrier of the vector-valued generalized function (8) in momentum space. Thus, under the proper choice of the functional domain of definition, the sum of the Wick series (2) satisfies all requirements of the pseudo-Wightman formalism including the generalized spectral condition which, in our opinion, may form a basis for a consistent Euclidean formulation of quantum field theory without positivity.

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