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# BOGOLIUBOV TRANSFORMATIONS IN THE THEORY OF PARAMETRIC RESONANCE FOR QUANTIZED FIELDS

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The time evolution of a quantized electromagnetic field appears to be a dynamical Bogoliubov transformation if the field is being pumped by the two-photon decays of Bose condensate particles.

In the classical work «On the Theory of Superfluidity» N.N.Bogoliubov for the first time wrote the special type of canonical transformations that mix creation and annihilation operators [1]:

$$\xi_f = \frac{b_f - L b^*_{-f}}{\sqrt{1-|L|^2}},\ \xi^*_f = \frac{b^*_f - L^* b_{-f}}{\sqrt{1-|L|^2}}. \tag{1}$$

Since then the Bogoliubov transformations are effectively used in statistical physics. I would like to present an example of dynamical Bogoliubov transformations that emerge in the case of a nonequilibrium system.

The system under consideration will consist of interacting electromagnetic and pseudoscalar fields with the phenomenological Lagrangian:

$$\mathcal{L} = \frac{\mathbf{E}^2 - \mathbf{H^2}}{2} + \frac{\dot\varphi^2 - (\vec\nabla\varphi)^2 - m^2\varphi^2}{2} + g\varphi\mathbf{E}\mathbf{H}. \tag{2}$$

Imposing the radiation gauge:

$$A_0 = 0,\ \mathrm{div}\mathbf{A} = 0,\ \mathbf{E} = -\dot{\mathbf{A}},\ \mathbf{H} = \mathrm{rot}\,\mathbf{A}, \tag{3}$$

and introducing the canonical momenta:

$$\Pi = \dot{\mathbf{A}} - g\varphi\mathbf{H},\ \pi = \dot\varphi, \tag{4}$$

we obtain the Hamiltonian density:

$$\mathcal{H} = \frac{(\Pi + g\varphi\mathbf{H})^2 + \mathbf{H^2}}{2} + \frac{\dot\varphi^2 + (\vec\nabla\varphi)^2 + m^2\varphi^2}{2}, \tag{5}$$

that consists of two positive contributions from each field. The interaction manifests through the noncommutativity of these two energy densities.

The canonical commutation relations (nontrivial) are:

$$[\pi(\mathbf{x},t),\ \varphi(\mathbf{x}',t)] \;=\; -i\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{6}$$

$$[\Pi_i(\mathbf{x},t),\ A_j(\mathbf{x}',t)] \;=\; -i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{x}') - i\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}.$$

They are consistent with the radiation gauge conditions:

$$\mathrm{div}\mathbf{A} = 0,\ \mathrm{div}\Pi = 0. \tag{7}$$

The second condition implies the negligible smallness of spatial variations of the $\varphi$ field and, in turn, simplifies further calculations.

With the help of the canonical commutators we obtain the rate of the energy exchange between the two fields:

$$\frac{d}{dt}\int d^3\mathbf{x}\frac{\mathbf{E}^2+\mathbf{H}^2}{2} = i\left[H_{tot},\ \int d^3\mathbf{x}\frac{\mathbf{E}^2+\mathbf{H}^2}{2}\right] =$$

$$= g\int d^3\mathbf{x}\frac{\pi\mathbf{HE}+\mathbf{EH}\ \pi}{2} = -\frac{d}{dt}\int d^3\mathbf{x}\frac{\pi^2+(\vec{\nabla}\varphi)^2+m^2\varphi^2}{2}. \tag{8}$$

This equation shows that the flow of energy between the two fields oscillates with a characteristic frequency of order of $\tau^{-1} = \langle g\pi\rangle$ if the system remains closed and conservative.

For a real open system in a thermal environment the rate of energy exchange can be enhanced through the Bose–Einstein condensation of the $\varphi$ field.

We have to consider this possibility in detail.

The canonical variables $\varphi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ can be represented by a plain wave decomposition:

$$\varphi(\mathbf{x},t) = \sum_{\mathbf{p}}\frac{1}{\sqrt{2VE_{\mathbf{p}}}}\left(b(\mathbf{p},t)\ e^{i\mathbf{px}} + b^\dagger(\mathbf{p},t)\ e^{-i\mathbf{px}}\right), \tag{9}$$

$$\pi(\mathbf{x},t) = \sum_{\mathbf{p}}\sqrt{\frac{E_{\mathbf{p}}}{2V}}\left(-ib(\mathbf{p},t)\ e^{i\mathbf{px}} + ib^\dagger(\mathbf{p},t)\ e^{-i\mathbf{px}}\right), \tag{10}$$

$$\sum_{\mathbf{p}}\cdots \equiv \int\frac{Vd^3\mathbf{p}}{(2\pi)^3}\cdots,\ E_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2},$$

$$\left[b(\mathbf{p},t),\ b^\dagger(\mathbf{p}',t)\right] = \frac{(2\pi)^3}{V}\delta^{(3)}(\mathbf{p}-\mathbf{p}'). \tag{11}$$

For the electromagnetic field the decomposition into circularly polarized plain waves is:

$$\mathbf{A}(\mathbf{x},t)=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega}}\left[\left(a^R(\mathbf{k},t)\ \mathbf{e}(\mathbf{k})+a^L(\mathbf{k},t)\ \mathbf{e}^*(\mathbf{k})\right)e^{i\mathbf{kx}}+H.c.\right], \quad (12)$$

$$\omega=|\mathbf{k}|,\ \mathbf{e}=\frac{1}{\sqrt{2}}\left(\mathbf{e}_\perp+i\frac{\mathbf{k}}{\omega}\times\mathbf{e}_\perp\right),\ \mathbf{e}_\perp\cdot\mathbf{k}=0,\ \mathbf{e}_\perp^2=1,$$

$$\Pi(\mathbf{x},t)=\sum_{\mathbf{k}}\sqrt{\frac{\omega}{2V}}\left[-i\left(a^R(\mathbf{k},t)\ \mathbf{e}(\mathbf{k})+a^L(\mathbf{k},t)\ \mathbf{e}^*(\mathbf{k})\right)e^{i\mathbf{kx}}+H.c.\right], \quad (13)$$

$$\mathbf{H}(\mathbf{x},t)=\sum_{\mathbf{k}}\sqrt{\frac{\omega}{2V}}\left[\left(a^R(\mathbf{k},t)\ \mathbf{e}(\mathbf{k})-a^L(\mathbf{k},t)\ \mathbf{e}^*(\mathbf{k})\right)e^{i\mathbf{kx}}+H.c.\right], \quad (14)$$

$$\left[a^R(\mathbf{k},t),\ a^{\dagger R}(\mathbf{k}',t)\right]=\left[a^L\mathbf{k},t),\ a^{\dagger L}(\mathbf{k}',t)\right]=\frac{(2\pi)^3}{V}\delta^{(3)}(\mathbf{k}-\mathbf{k}'). \quad (15)$$

If we, for physical reasons, only take the condensate mode into account $\varphi(\mathbf{x},t)\equiv\varphi(t)$:

$$\varphi(t)=\frac{1}{\sqrt{2mV}}\left(b(t)+b^\dagger(t)\right),\ \pi(t)=\sqrt{\frac{m}{2V}}\left(-ib(t)+ib^\dagger(t)\right), \quad (16)$$

$$\left[b(t),\ b^\dagger(t)\right]=1,\ [\pi(t),\ \varphi(t)]=-\frac{i}{V}, \quad (17)$$

the total Hamiltonian becomes:

$$H=\int_V d^3\mathbf{x}\frac{\Pi^2+\mathbf{H}^2}{2}+V\frac{\pi^2+m^2\varphi^2}{2}+g\varphi\int_V d^3\mathbf{x}\frac{\Pi\mathbf{H}+\mathbf{H}\Pi}{2}+g^2\varphi^2\int_V d^3\mathbf{x}\frac{\mathbf{H}^2}{2}. \quad (18)$$

For reasonable fields $g^2<\varphi^2>\ll 1$, thus the last term in the equation above is negligible. Without it the total Hamiltonian takes the following momentum representation form:

$$H=m\left(b^\dagger(t)\ b(t)+\frac{1}{2}\right)+$$
$$\sum_{\mathbf{k}}\omega\left(a^{\dagger R}(\mathbf{k},t)\ a^R(\mathbf{k},t)+a^{\dagger L}(\mathbf{k},t)\ a^L(\mathbf{k},t)+1\right)+$$
$$\frac{g}{\sqrt{2mV}}\left(b(t)+b^\dagger(t)\right)\sum_{\mathbf{k}}\frac{\omega}{2}\left[-i\left(a^R(\mathbf{k},t)\ a^R(-\mathbf{k},t)-\right.\right.$$
$$\left.\left.a^L(\mathbf{k},t)\ a^L(-\mathbf{k},t)\right)+H.c.\right]. \quad (19)$$

It is convenient to separate the high frequency time dependence by going over to the slow varying tilde operators:

$$a^R(\mathbf{k},t) = e^{-i\omega t}\,\tilde{a}^R(\mathbf{k},t), \dots,\ b(t) = e^{-imt}\,\tilde{b}(t) \dots . \tag{20}$$

Their time dependence is determined by the interaction Hamiltonian

$$\dot{\tilde{b}}(t) = i\left[H_{int}(t),\ \tilde{b}(t)\right], \dots, \dot{\tilde{a}}^R(\mathbf{k},t) = i\left[H_{int}(t),\ \tilde{a}^R(\mathbf{k},t)\right], \dots, \tag{21}$$

in which we retain only the resonant $\omega = m/2$ terms:

$$H_{int}(t) = \frac{g}{\sqrt{2mV}} \sum_{\mathbf{k}} \frac{\omega}{2} \times$$

$$\times \left[-i\tilde{b}^\dagger(t)\left(\tilde{a}^R(\mathbf{k},t)\tilde{a}^R(-\mathbf{k},t) - \tilde{a}^L(\mathbf{k},t)\tilde{a}^L(-\mathbf{k},t)\right) + H.c.\right] \tag{22}$$

and omit the fast varying nonresonant terms like $\tilde{a}\;\tilde{a}\;\tilde{b}\;e^{-i(m+2\omega)t}$.

For every fixed direction and circular polarization of a plain electromagnetic wave there are two coupled equations, e.g.:

$$\dot{\tilde{a}}^R(\mathbf{k},t) = \frac{g\omega}{\sqrt{2mV}}\,\tilde{b}(t)\,\tilde{a}^{\dagger R}(-\mathbf{k},t),\ \ \dot{a}^{\dagger R}(-\mathbf{k},t) = \frac{g\omega}{\sqrt{2mV}}\,\tilde{b}^\dagger(t)\,\tilde{a}^R(\mathbf{k},t). \tag{23}$$

Let the decay of the Bose condensate be compensated by a new delivery to it. In this case we may consider $b^\dagger b = N \gg 1$ as a constant $c$-number, so the last equations become linear and easily solvable. We have come to the dynamical Bogoliubov transformation:

$$\tilde{a}^R(\mathbf{k},t) = \mathrm{ch}\left(\frac{g}{2}\sqrt{\frac{mN}{2V}}\;t\right)\tilde{a}^R(\mathbf{k},0) + \mathrm{sh}\left(\frac{g}{2}\sqrt{\frac{mN}{2V}}\;t\right)\frac{b}{\sqrt{N}}\,a^{\dagger R}(-\mathbf{k},0). \tag{24}$$

The considered phenomenon is a clear case of quantum parametric resonance [2]. The given condensate field plays the role of the time dependent external field that pumps the coherent electromagnetic field. The application of the obtained result to the feasibility study of positronium gamma laser was done in the author's work [3]. The coherent creation of scalar boson pairs by a time dependent external electric field is also described by a dynamical Bogoliubov transformation [4].

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