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SYMMETRY AND ANALYTICITY

M.P.Chavleishvili

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
141980 Dubna, Russia and International University «Dubna»

On the basis of general space-time and crossing symmetry, a general analytic structure for amplitudes describing spin-particle binary reactions is considered. Using knowledge about the kinematic structure of helicity amplitudes in the dynamic amplitude approach we can get: dispersion relations for helicity amplitudes; low-energy theorems; sum rules; model-independent sum-rule type inequalities for observable quantities and some asymptotic relations between polarization parameters. In this short paper we will consider only dispersion relations for each individual helicity amplitudes describing any elastic processes.

1. INTRODUCTION

A nice analytic structure of the scattering amplitude can be seen from dispersion relations. Dispersion relations for pion-nuclon scattering for fixed t were first proved by Nikolai Nikolaevich Bogoliubov in axiomatic approach to quantum field theory. This famous work was presented at the Siettle conference in 1956 (see [1]). In the framework of the so-called S-matrix approach [2], dispersion relations are postulated — they are considered as basis of theory.

For real binary processes with particles of nonzero spins, analytic structures of amplitudes are defined by spin-kinematics (they give us kinematic singularities) and general properties defined by the unitarity condition (dynamic singularities).

On the basis of general space-time and crossing symmetry, a general analytic structure for amplitudes describing spin-particle binary reactions is considered. Using knowledge about the kinematic structure of helicity amplitudes in the dynamic amplitude approach we can get: dispersion relations for helicity amplitudes; low-energy theorems; sum rules; model-independent sum-rule type inequalities for observable quantities and some asymptotic relations between polarization parameters. In this short paper we will consider only dispersion relations for each individual helicity amplitudes describing any elastic processes.

2. SYMMETRY AND SPIN PARTICLES

Due to the symmetry in particle physics (quantum field theory), we have a Lagrangian of a definite form that depends on a certain number of masses and interaction constants. This is in sharp contrast with quantum mechanics where interactions are considered as arbitrary functions (potentials) for every pair of particles. The symmetry does not admit arbitrary functions.

Today we have the following succession:

$$\text{Symmetry} \rightarrow \text{group} \rightarrow \text{particle interaction.}$$

Besides the Lagrangian approach, in particle physics there exists a problem that has its own history: the problem of direct investigation of processes with elementary particles, based on the general principles and independent of the explicit form of Lagrangian. This general principles are: symmetry, causality and unitarity.

Analytic properties of the amplitudes of certain particle reaction are connected with causality and unitarity, and properly defined amplitudes obey dispersion relations. Dispersion relations for invariant amplitudes of pion-nucleon interaction were given in [1].

In studying analytic properties of amplitudes, we have two types of singularities: dynamic singularities connected with unitarity and kinematic singularities connected with spin.

3. SPIN AND PARTICLE REACTIONS

Most of the particles have a nonzero spin. We are going to consider binary reactions with particles of arbitrary spins. If we consider reactions with particles with spin

$$s_1 + s_2 \rightarrow s_3 + s_4, \tag{1}$$

we have $N = (2s_1 + 1)(2s_2 + 1)(2s_3 + 1)(2s_4 + 1)$ functions to describe the process, and we must choose the optimal set of these functions. The spin-particle reactions are convenient to describe in the helicity amplitude formalism [3]. Helicity amplitudes $f_{\lambda_3,\lambda_4;\lambda_1,\lambda_2}(s,t)$ have a clear physical meaning, observables are expressed by them in a simple way. Helicity amplitudes contain all the information about the considered process. Helicity amplitudes have kinematic singularities independent of interactions.

Scattering of spinless particles is described by one amplitude. Considering this amplitude as a function of invariant variables, we have the function $A(s,t)$. This amplitude has some singularities. They are called the dynamic singularities.

For spin-particles, the process is described by several functions, several helicity amplitudes. And they have additional, so-called kinematic, singularities. So helicity amplitudes do not fulfil simple dispersion relations. It is necessary to find and separate kinematic singularities. So, helicity amplitudes are expressed via a set of other amplitudes without kinematic singularities. For a lowest spin it is convenient to introduce invariant amplitudes.

Let us consider the simplest nontrivial reaction: π-N scattering, elastic scattering of a spin-zero particle with the mass μ on the spin-1/2 particle of mass m. Using the Dirac equation one can find the following connection between the helicity and invariant amplitudes (in the standard notation):

$$f^s_{0,\lambda_4;0,\lambda_2}(s,t) = \bar{u}^{\lambda 4}(p_4)\{A(s,t) + \hat{Q}B(s,t)\}u^{\lambda 2}(p_2). \tag{2}$$

Here $A(s,t)$ and $B(s,t)$ are invariant amplitudes. Properly defined invariant amplitudes have no kinematic singularities.

For the general case of scattering of particles with spins s_i we have relations of the following type [4]:

$$f_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s,t) = \sum_{n=1}^{N} a^n_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s,t)A_n(s,t). \tag{3}$$

Kinematic singularities of $f_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s,t)$ are contained in the coefficient functions $a^n(s,t)$.

This procedure is nice for low spins. It is difficult to construct such an expansion for high spins; for all $s_i = 3/2$, $N = 256$ and for $s_i = 11/2$, $N \simeq 20000$. Besides, the main difficulty is to find a decomposition of that type so that coefficients of invariant amplitudes do not contain «secret singularities» rather than in dimensions. So, in describing the Compton effect for several years people used a decomposition suggested in [5], but then it appeared that those invariant amplitudes had additional singularities, and later a more complicated decomposition [6] was suggested.

Besides technical difficulties for spins larger than 1, a nontrivial question of uniqueness of that decomposition arises, and since for higher spins the invariant amplitude decomposition is not unique, «secret» singularities, additional and noncontrollable kinematic constraints appear.

There exists another way based on symmetry principles, and it uses representations of a rotation group — Wigner's d-functions. If we use d-functions in the s-channel, then use d-functions in the t-channel, and finally connect channels also by d-functions, we can get a result much more convenient than (3).

4. ANALYTIC PROPERTIES OF AMPLITUDES: SEPARATION OF SPIN-KINEMATICS AND DYNAMICS

A lot of people worked in this direction by considering spin-kinematics and decomposition of helicity amplitudes in terms of other sets of amplitudes [4–6].

Combining some approaches and modifying others we suggest a new variant of the formalism that has all advantages of different approaches, differs from all of them, and is based on the symmetry and conservation laws, and is general and simple.

Symmetry imposes restrictions on amplitudes. When one has additional symmetries in definite directions, the number of independent amplitudes in such «symmetric directions» is reduced. Such situations occur for forward and backward scattering.

Consider the reaction in the s-channel described by the helicity amplitudes. Introduce the quantities $\lambda = \lambda_1 - \lambda_2$ and $\mu = \lambda_3 - \lambda_4$. Two particles in the centre-of-mass system are moving in the opposite directions, and thus, λ and μ are projections of the total spin in the directions of motion prior to and after collision. Owing to the conservation of the projection of the total angular momentum, the amplitudes in the forward direction, $\theta_s \to 0$, should vanish in all cases except for $\lambda = \mu$. Analogously, for backward scattering, $\theta_s \to \pi$, the amplitudes should vanish for the same reasons in all cases except for $\lambda = -\mu$.

For forward scattering we have

$$f^{\text{forward}}_{\lambda_3\lambda_4,\lambda_1,\lambda_2} = \begin{cases} f_{\lambda_3\lambda_4,\lambda_1,\lambda_2}, & \text{when} \quad \lambda = \mu, \\ 0, & \text{when} \quad \lambda \neq \mu, \end{cases} \tag{4}$$

whereas for backward scattering

$$f^{\text{backward}}_{\lambda_3\lambda_4,\lambda_1,\lambda_2} = \begin{cases} f_{\lambda_3\lambda_4,\lambda_1,\lambda_2}, & \text{when} \quad \lambda = -\mu, \\ 0, & \text{when} \quad \lambda \neq -\mu. \end{cases} \tag{5}$$

Two questions arise: Can the helicity amplitudes be parametrized so as to satisfy the conditions (4) and (5) automatically? Can kinematic singularities of helicity amplitudes be found and separated in a simple way? The answer to both questions is «yes».

For the spinless case we have the decomposition via the Legendre polynomials depending on $\cos\theta$. By definition, in the spinless case we have no kinematic singularities.

In the nonzero spin case, helicity amplitudes have decomposition via Wigner d-functions of rotation. Helicity amplitudes are splitted into two parts; one part is defined by the symmetry properties and enters into the functions $d^J_{\lambda\mu}(\cos\theta)$ [7] that make the conservation laws of the angular momentum valid, and the other part has a dynamic nature and enters into the partial helicity amplitudes $f^J_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s)$.

Kinematic singularities in d-functions do not depend on J, and we can separate the common singular factors. The rest sum in the decomposition contains the decomposition over polynomials in the t-variable. So, we can defin the so-called dispersion amplitudes [8] for any binary processes:

$$f^s_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s,t) = A^{|\lambda-\mu|}B^{|\lambda+\mu|}\bar{f}^s_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s,t), \quad (6)$$

here

$$A = \frac{\sqrt{L^2-a^2}}{(m_1+m_2)(m_3+m_4)}, B = \frac{\sqrt{L^2+a^2}}{(m_1+m_2)(m_3+m_4)},$$

$$\begin{aligned} L^2 \quad = \quad & \{[s-(m_1+m_2)][s-(m_1+m_2)] \\ & [s-(m_3+m_4)][s-(m_3+m_4)]\}^{1/2}, \end{aligned}$$

$$a^2 = 2st + s^2 - s\sum m_k^2 + (m_1^2-m_2^2)(m_3^2-m_4^2).$$

The mass factors in the denominators make A and B dimensionless without introducing additional singularities in the variable s . Under this parametrization, the conditions (4) and (5) are fulfilled automatically. All kinematic singularities in variable t are separated explicitly, and no false singularities in s are introduced. The amplitudes $\bar{f}^s_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s,t)$ suit well for studying the analytic properties of the amplitudes at fixed s, because they obey dispersion relations. Therefore, we call them the dispersion amplitudes [12]. They still may have the kinematic singularities in the variable s.

Dispersion amplitudes remind reduced amplitudes [10], but they have no additional s-variable false singularities.

For t-channel processes the corresponding dispersion amplitudes are free from kinematic singularities in the variable s. Expressing the dispersion amplitudes of the s-channel in terms of the dispersion amplitudes on the annihilation channel, we obtain the connection between the amplitudes having kinematic singularities in s with the amplitudes that are free from them. So, kinematic singularities of the s-channel helicity amplitudes are in crossing coefficients in crossing relations between s- and t-channel amplitudes. The number of coefficients is restricted, and we do know the singularities of these coefficients; indeed these coefficients are Wigner's functions, and we do know their singularities!

So, using crossing symmetry we can find kinematic singularities of the s-channel dispersion amplitudes also in the variable s; separating these singularities we determine a new set of functions describing binary processes — dynamic amplitudes. Dynamic amplitudes for elastic processes ($m+\mu \longrightarrow m+\mu$) have the following relations with the helicity amplitudes [11]:

$$f_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s,t) = \left(\frac{\sqrt{-t}}{m+\mu}\right)^{-|\lambda-\mu|}\left(\frac{\sqrt{L^2+st}}{(m+\mu)^2}\right)^{-|\lambda+\mu|} \times$$
$$\times\left(\frac{L}{(m+\mu)^2}\right)^{-2(s_1+s_2)} D_{\lambda_3\lambda_4,\lambda_1,\lambda_2}(s,t). \qquad (7)$$

Dynamic amplitudes are in fact modified regularized helicity amplitudes, they differ from the reduced amplitudes by dimensions: all dynamic amplitudes have the same dimensions, whereas the dimensions of regularized amplitudes depend on spins and helicities [10].

5. ANALYTIC PROPERTIES AND DISPERSION RELATIONS FOR INDIVIDUAL HELICITY AMPLITUDES

Let us consider dispersion relations with fixed t, a certain number of them was strongly proved for definite regions of t, and which are used much more frequently, then relations with fixed s. For getting such dispersion relations one has to have amplitudes free of kinematic singularities in s and u variables. Such functions are: dynamic amplitudes, correctly defined invariant amplitudes, and t-channel dispersion amplitudes. Of course, considering process in the centre-of-mass system of s-channel it is convenient to use s-channel amplitudes.

Dynamic amplitudes when t is fixed fulfil the following dispersion relations:

$$D_h(s,t) = D_h^B(s,t) + \frac{1}{\pi}\int\limits_{s_0}^{\infty}\frac{ds'}{s'-s}\Big\{D_h(s',t)\Big\}^s + \frac{1}{\pi}\int\limits_{u_0}^{\infty}\frac{du'}{u'-u}\Big\{D_h(u',t)\Big\}^u. \qquad (8)$$

One can easily add corresponding subtraction terms, if they are necessary.

Taking into account a simple, one-to-one correspondence between dynamic and helicity amplitudes, we get dispersion relations for each individual helicity amplitudes for any spin-particle elastic scattering:

$$f_h(s,t) = f_h^B(s,t) + K_h(s,t) \times$$
$$\times\left\{\frac{1}{\pi}\int\limits_{s_0}^{\infty}\frac{ds'}{s'-s}\left\{\frac{f_h(s',t)}{K_h(s',t)}\right\}^s + \frac{1}{\pi}\int\limits_{u_0}^{\infty}\frac{du'}{u'-u}\left\{\frac{f_h(u',t)}{K_h(u',t)}\right\}^u\right\}. \qquad (9)$$

In invariant amplitudes it is possible to get dispersion relations for combinations of helicity amplitudes if the connection matrix between helicity and invariant amplitudes is known. But they are known only for small values of spins, and even in this case they are very complicated.

6. OTHER APPLICATIONS OF DYNAMIC AMPLITUDES

The spin kinematics allows one to obtain the low-energy theorems for photon-hadron processes [12] and gravitino scattering on a spin-0 target. For the latter process at low energies, the helicity amplitudes up to $0(E^3)$ are determined by their t-channel Born terms with the photon exchange [13]. The dynamic amplitudes, or more simply the t-channel dispersion amplitudes, can be used to prove model-independent dispersion inequalities for the Compton effect on a pion and a nucleon target, including the case of the polarized photon scattering [14].

In the framework of the «dynamic amplitude» approach, obligatory kinematic factors arise in the expressions of observables. These spin structures for high energies give a small parameter that orders the contributions of helicity amplitudes to observables. Such a «kinematic hierarchy» predicts a simple connection between asymmetry parameters and even numerical values for them [15] for pp elastic scattering at high energies and a large fixed angle $(90°)$.

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