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UNITARY TRANSFORMATIONS IN QUANTUM FIELD THEORY AND BOUND STATES

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Finding the eigenstates of the total Hamiltonian H or its diagonalization is the important problem of quantum physics. However, in relativistic quantum field theory (RQFT) its complete and exact solution is possible for a few simple models only. Unitary transformations (UT's) considered in this survey do not diagonalize H, but convert H into a form which enables us to find approximately some H eigenstates. During the last years there have appeared many papers devoted to physical applications of such UT's. Our aim is to present a systematic and self-sufficient exposition of the UT method. The two general kinds of UT's are pointed out, distinct variations of each kind being possible. We consider in detail the problem of finding the simplest H eigenstates for interacting mesons and nucleons using the so–called «clothing» UT and Okubo's UT. These UT's allow us to suggest definite approaches to the problem of two-particle (deuteron-like) bound states in RQFT. The approaches are shown to yield the same two-nucleon quasipotentials in the first nonvanishing approximation. We demonstrate how the particle mass renormalization can be fulfilled in the framework of the «clothing» procedure. Besides the UT of the Hamiltonian we discuss the accompanying UT of the Lorentz boost generators.

Важной проблемой квантовой физики является нахождение собственных состояний полного гамильтониана H или его диагонализация. В релятивистской квантовой теории поля (РКТП) полное и точное решение этой задачи возможно только для немногих простых моделей. Рассматриваемые в обзоре унитарные преобразования (УП) не диагонализуют H, они приводят H к виду, позволяющему приближенно найти только некоторые собственные состояния и значения H. За последние годы появилось много работ по физическим приложениям таких УП. В обзоре дано систематическое изложение метода УП. Указываются два главных вида УП с возможными вариациями каждого вида. Детально рассматривается задача нахождения простейших собственных состояний H для юкавского взаимодействия нуклонов и мезонов с помощью так называемого «одевающего» УП и УП Окубо. Эти преобразования позволяют предложить два подхода к решению проблемы двухчастичных (дейтроноподобных) связанных состояний в РКТП. Показано, что эти подходы в первом приближении дают одинаковые двухнуклонные квазипотенциалы. Демонстрируется, как с помощью «одевающего» УП осуществляется перенормировка массы частицы. Кроме УП гамильтониана обсуждается УП генераторов лоренцевских бустов.

1. INTRODUCTION

The so-called unitary transformation (UT) method has the same age as the quantum theory itself. Its first applications for constructing Hermitian effective

interactions (HEI) can be found in [1] and [2] (in this connection, see a review article [3] where the different perturbation expansions of HEI were discussed). In quantum field theory the first considerations using the UT method were given by Wentzel [4] and Heitler [5].

A number of the schemes for reduction of the exact eigenvalue problem to the model–space problem via the various UT's were put forward in 50's ([6–8]) during the extensive development of the meson theory of nuclear forces (see, e.g., [9]). Owing to the work [10] this approach has proved to be very useful in studies of electromagnetic (e.m.) interactions with nuclei for nonmesonic channels (in particular, when constructing the effective operators of meson exchange currents (see, e.g., [11], surveys [12,13] and refs. therein).

Along with this guideline the UT method was used for the formulations of RQFT in terms of physical or «clothed» particles ([14–18], see also our talks at the recent conferences [19] and [20]).

By using the Okubo UT the authors of [21] constructed effective generators for the Poincaré algebra, acting on nucleonic degrees of freedom only. It was made in the framework of perturbation theory for a simple model of «spinless» nucleons exchanging scalar mesons. Therefore, we have an instructive example of how some noncommuting Hermitian operators can be reduced by one and the same UT to Okubo's block form.

Late 80's and 90's have brought a renewed interest to this area. First of all, we mean applications [22,23] of the UT method to a covariant treatment of the two-body bound-state problem (cf. [24]). The corresponding transformed Hamiltonian and boost operators do not couple (in the second order in meson-nucleon coupling constants) the nucleon (no-meson) subspace with its complement in the full Fock space of hadron states. In the work [25] the same method has been employed to derive the effective nucleon-nucleon and nucleon–antinucleon interactions starting from a field Hamiltonian with the exchange of π, ρ, ω and, σ mesons. Then, within the Hartree approximation, these effective interactions have been introduced for describing the saturation properties of nuclear matter. Note also recent explorations [26, 27] of the hard–core problem in the theory of nuclear forces.

In papers [28,29] some extensions of the UT method have been suggested for constructing effective current operators in the theory of photomeson processes on nuclei (see also Ref. 30 where one can find the calculations of the four structure functions for pion electroproduction on the deuteron near threshold). Recently, the UT method has been used [31] for deriving effective two-particle one-meson exchange potentials in the instant and front forms of relativistic quantum mechanics. At last, with the aid of a modification of the method the authors of [32] have proposed a meson-exchange model for πN scattering and γN-πN reaction. Certainly, one may say that nowadays the UT method has survived its second birth.

This review is focused upon an application of UT's in RQFT and aimed at an approximate treatment of the physical vacuum, the observable one-particle and two-particle bound and scattering states. These states have a common feature, viz., they do not change in time. The simplest example is the state without observable particles, i.e., the physical vacuum. Other examples give free particles with a definite four-momentum (elementary particles, atoms and nuclei in their ground states).

Our point of departure is that such states should be eigenvectors of a Hamiltonian H, which are stationary. In the context, the «bare» states, i.e., eigenvectors of a free part H_0 of the total Hamiltonian H, are inappropriate for theoretical description of the physical objects. Firstly, they change in time although the possible transitions to other states are virtual and can be considered undetectable because of their small probabilities. Secondly, H_0 has no eigenvectors which would correspond to bound states (e.g., the hydrogen atom or the deuteron).

From this postulate there follows such a definition for a bound state of the deuteron-type in RQFT: it should be described by a proper eigenvector of the total Hamiltonian of the theory. In other words, the deuteron problem should be reduced to an exact or approximate solution of the H eigenvalue problem. There are various approaches to this problem, viz., within the Bethe–Salpeter formalism, via its three-dimensional versions, etc. (see, e.g., [33,34] and refs. therein). However, the links between the respective «wavefunctions» (for instance, the Bethe–Salpeter amplitudes for bound states) and the exact or approximate H eigenvectors are very sophisticated. According to our postulate similar links in the framework of the approach developed below are much more direct and transparent (see Sec. 4). In addition, the description of particle scattering can be reduced to calculation of the relevant scattering wavefunctions.

The reasons for employing just RQFT when describing bound and scattering states are well known. Being not satisfied with the multitude of disconnected phenomenological explanations we strive for a unified description of Nature. RQFT's are the best known candidates for unified theories. Firstly, they give a qualitative and natural consideration of particle creation and destruction. Secondly, local RQFT's ensure in a sense the relativistic causality unlike phenomenological approaches.

General idea of constructing the UT's in question can be formulated in the following way.

Any UT of H can be considered as a transformation expressing the H-matrix determined with respect to a new basis in the Hilbert (Fock) space of a given physical system through the H-matrix with respect to the old basis in the same space (details see in Sec. 6). Normally, the latter is composed of eigenvectors (the «bare» states) of the free part H_0 in the partition $H = H_0 + V$, where V represents the interaction between the fields involved. In general, the new H-matrix turns

out to be more complicated. For example, in the case with Yukawa coupling $V \sim \int \bar{\psi}(\mathbf{x})\gamma_5\psi(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}$, that conserves the baryon number and may change the meson number merely by unit, nonzero elements of the initial H-matrix are either diagonal, or near diagonal (e.g., they can be of the $\langle N' \mid H \mid \pi N\rangle$-kind). In a new representation for this matrix all its elements can be nonzero (e.g., along with the aforementioned elements one can meet the elements $(N'N' \mid H \mid NN)$), $(\pi'N' \mid H \mid \pi N)$, $(N' \mid H \mid \pi\pi N)$, and the others in which the meson number can be arbitrarily altered*). The elements $(N'N' \mid H \mid NN)$ contribute to the NN scattering even in the first order of perturbation theory. Of course, there are other elements of the transformed H-matrix, which contribute to the NN scattering as well, but in higher orders.

Let us impose the following constraint upon the transformation associated with the basis change: in the new H-matrix the NN interaction should be described only through the elements $(N'N' \mid H \mid NN)$, i.e., all the other elements which could contribute to the NN scattering should be zero. This means that the H-matrix must be reduced to a block diagonal form where specific off-diagonal blocks consist of zero elements (other clarifications can be found in Sec. 6). This requirement (the Okubo condition [7]) can be replaced by the other constraints, more flexible and easier realizable.

Our aim is to present a systematic exposition of the UT method. A main attention within the method is paid to the two approaches, viz., the clothing procedure and the blockdiagonalization after Okubo. We strive for a self-sufficient presentation which may be understood without referring to original papers. As a rule, we avoid to point out mistakes and obscurities in the latter (but sometimes allow ourselves to note their tacit assumptions).

This review is organized as follows.

In Sec. 2 we consider the problem of obtaining the simplest H eigenstates. The lowest H eigenstate Ω can be juxtaposed to the state without observable particles (the physical vacuum). Further, we seek one-particle-like H eigenstates which have «bare» partners (e.g., in the case of the interacting pion and nucleon fields the «bare» one-meson $a^\dagger(\mathbf{k})\Omega_0$ and one-nucleon $b^\dagger(\mathbf{p})\Omega_0$ states, Ω_0 being the «bare» vacuum). These H eigenstates may be called «clothed» [14] because meson-nucleon interaction is taken into account when constructing such states. At this point, instead of the usual «bare» creation-destruction operators $a^\dagger(\mathbf{k})$, $a(\mathbf{k})$, ... we introduce the «clothed» operators ${a_c}^\dagger(\mathbf{k})$, $a_c(\mathbf{k})$, ..., so that the physical one-meson state is described by the vector ${a_c}^\dagger(\mathbf{k})\Omega$.

While finding the above H eigenstates one has to determine a «clothing» UT such that the transformed Hamiltonian does not contain the interaction terms which

*In order to distinguish the matrix elements with respect to the new basis we employ round brackets.

correspond to some virtual energy-nonconserving processes (e.g., $N \to \pi N$, $\pi \to N\bar{N}$). These terms are called «bad».

As one might expect the mass of the clothed particle turns out to be unequal to the respective bare mass. The related problem of mass renormalization is explored in Subsec. 2.5.

The clothing transformation of the Hamiltonian results in its representation through the clothed creation–destruction operators. A basic task of Sec. 3 is to investigate how transformed generators of the Lorentz boosts depend on the clothed operators. We show that the bad terms can be removed via the same clothing transformation simultaneously from the total Hamiltonian and these generators. A consequence of such reduction is that the «clothed» vacuum Ω remains invariant under Lorentz transformations (unlike the bare vacuum Ω_0) while the clothed one-particle states have the proper transformation properties (in particular, their momenta being suitably changed).

Along with the clothed one-particle states the total Hamiltonian can have one-particle-like states (e.g., bound states of the deuteron-type) without any bare partners. In Sec. 4 within our clothing procedure we suggest an approximate way of finding such states. The resulting bound-state equations resemble the Schroedinger equation for stationary states in nonrelativistic quantum mechanics, where the usual potentials are replaced by the interactions between clothed particles (the quasi-potentials). Explicit analytical expressions for the model nucleon-nucleon and meson-nucleon quasipotentials are given in Subsec. 4.3.

In Sec. 5 we discuss some modifications and extensions of the clothing approach.

Section 6 starts with a definition of the similarity transformation $H \to U^{-1}HU$ ($U^{-1} = U^{\dagger}$) as a transformation of the matrix $\langle n' \mid H \mid n\rangle$ (determined with respect to a set of orthonormal vectors $\mid n\rangle$) into the matrix $(\nu' \mid H \mid \nu)$ with respect to another basis $\mid \nu)$.

The Okubo condition on U and the corresponding decoupling equation from [7] are presented in Subsec. 6.2.

Two operator kinds of the Okubo UT are considered in the framework of the Okubo approach using the new particle creation–destruction operators which correspond to the clothed operators determined in Sec. 2. This allows us to establish the relation to the clothing procedure and its modifications (cf. Secs. 2 and 5).

Some original results of this work are summarized in Sec. 7.

More technical details and some auxiliary calculations are referred to Appendix A. Appendix B is devoted to some mathematical aspects related to the UT method. We suggest an algebraic approach which enables one to treat the clothing UT as an element of an algebra lacking any operator representation. At last, Appendix C exemplifies an explicit solution of Okubo's decoupling equation for a simplified field model.

2. CLOTHED PARTICLES IN QUANTUM FIELD THEORY

The notion of clothed particles will be considered using the following model: a spinor (fermion) field ψ interacts with a neutral pseudoscalar meson field ϕ by means of the Yukawa coupling. The model Hamiltonian is $H = H_0 + V$, where

$$H_0 = \int \bar{\psi}(\mathbf{x})[-i\boldsymbol{\gamma}\boldsymbol{\nabla} + m_0]\psi(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int \left[\pi^2(\mathbf{x}) + (\boldsymbol{\nabla}\phi(\mathbf{x}))^2 + {\mu_0}^2\phi^2(\mathbf{x})\right] d\mathbf{x}, \tag{2.1}$$

$$V = ig\int \bar{\psi}(\mathbf{x})\gamma_5\psi(\mathbf{x})\phi(\mathbf{x})d\mathbf{x}. \tag{2.2}$$

For simplicity, we do not employ a more refined form of H properly symmetrized in the fields involved (see, e.g., [15,37]). This model has much in common with more realistic models for the interacting fields (e.g., the nucleon isodoublet (p,n) interacting with the meson isotriplet (π^+, π^0, π^-).

The Hamiltonian can be expressed in terms of bare destruction (creation) operators $a(\boldsymbol{k})$ $(a^\dagger(\boldsymbol{k}))$, $b(\boldsymbol{p}, r)$ $(b^\dagger(\boldsymbol{p}, r))$ and $d(\boldsymbol{p}, r)$ $(d^\dagger(\boldsymbol{p}, r))$ of the meson, the fermion and the antifermion (see Eqs. (2.8) and (2.16)). Here $\boldsymbol{k}$ and $\boldsymbol{p}$ denote the momenta; r is the spin index. An exact definition of the bare operators which we use will be given in Subsec. 2.2. In what follows, the set of all these operators is denoted by a symbol a, while a_p is used for one of them. The state without bare particles Ω_0 and the bare one-particle states $a^\dagger(\boldsymbol{k})\ \Omega_0$, $b^\dagger(\boldsymbol{p}, r)\ \Omega_0$ and $d^\dagger(\boldsymbol{p}, r)\ \Omega_0$ are not H eigenvectors.

2.1. Clothed Particle Operators and States. Now, we introduce new destruction (creation) operators

$$a_c(\mathbf{k})({a_c}^\dagger(\mathbf{k})),\ b_c(\mathbf{p}, r)({b_c}^\dagger(\mathbf{p}, r)) \text{ and } d_c(\mathbf{p}, r)({d_c}^\dagger(\mathbf{p}, r)) \quad \forall \mathbf{k}, \mathbf{p}, r \tag{2.3}$$

with the following properties:

i) The physical vacuum (the H lowest eigenstate) must coincide with a new no-particle state Ω, i.e., the state that obeys the equations

$$a_{\mathrm{c}}(\boldsymbol{k}) \mid \Omega\rangle = b_{\mathrm{c}}(\boldsymbol{p}, r) \mid \Omega\rangle = d_{\mathrm{c}}(\boldsymbol{p}, r) \mid \Omega\rangle = 0 \quad \forall \mathbf{k}, \mathbf{p}, r, \tag{2.4}$$

$$\langle\Omega \mid \Omega\rangle = 1.$$

ii) New one-particle states ${a_c}^\dagger(\mathbf{k})\Omega$, etc., are H eigenstates as well.

iii) The spectrum of indices that enumerate the new operators must be the same as that for the bare ones (this requirement has been used when writing Eq.(2.3)).

iv) The new operators satisfy the same commutation rules as their bare partners do. For instance,

$$[a_{\mathrm{c}}(\boldsymbol{k}), {a_{\mathrm{c}}}^\dagger(\boldsymbol{k'})] = \delta(\boldsymbol{k} - \boldsymbol{k'})\ ,$$

$$\left\{b_{\rm c}(\boldsymbol{p},r),b_{\rm c}{}^{\dagger}(\boldsymbol{p'},r')\right\}=\left\{d_{\rm c}(\boldsymbol{p},r),d_{\rm c}{}^{\dagger}(\boldsymbol{p'},r')\right\}=\delta_{rr'}\delta(\boldsymbol{k}-\boldsymbol{k'})\ . \qquad (2.5)$$

Following [14,15] we shall call clothed the new operators and states. Note that the name is sometimes used in a sense which differs from that defined by the points i) – iv).

As one can see, the problem of clothing is equivalent to determination of some H eigenvectors. In fact, the property iii) means that we do not pretend to find all H eigenstates which are one-particle-like. For example, H may have a deuteron-like eigenstate with a mass $< 2m$, where m is the nucleon mass. No bare one-particle state corresponds to such a state. Now we intend to find only those one-particle-like eigenstates of H which have bare partners.

One should stress that the clothing problem may turn out to be unsolvable. A solvability condition will be pointed out in Subsec. 2.4. Note also that the properties i) – iv) can be supplemented by some physical constraints which will be discussed in Sec. 3, but are not needed here.

Some clothing procedures have been realized within simple field models (see, e.g., [14, 15, 17]). In the paper, we use a kind of perturbation theory developed in [5, 16] and [18]. It can be applied to any field theory to yield an approximate solution of the problem.

2.2. Bare Particles with Physical Masses. By definition, the bare one-fermion eigenstate $\mid \mathbf{p},r\rangle_0$ of the operator H_0, being simultaneously the eigenstate of total momentum $\boldsymbol{P}$, belongs to the H_0 eigenvalue $E^0_{\mathbf{p}}=\sqrt{\mathbf{p}^2+{m_0}^2}$. Let us consider an H eigenstate $\mid \mathbf{p},r\rangle$ for which $\mid \mathbf{p},r\rangle_0$ is a zeroth approximation (ZA). Perturbation theory shows that the corresponding H eigenvalue $E_{\mathbf{p}}$ differs from $E^0_{\mathbf{p}}$. In the relativistic case the function $E_{\mathbf{p}}$ must be of the form $\sqrt{\mathbf{p}^2+m^2}$ where m is the mass of an observed free fermion. We shall call it physical mass. Analogously, one can argue appearance of the meson physical mass μ which differs from the trial mass μ_0.

So, we expect that the physical fermion and meson masses m and μ arise in a natural manner when finding H eigenvalues which correspond to the clothed one-particle states.

Such an introduction of the masses m and μ can be used to divide the total Hamiltonian into the new free part H_F and the new interaction H_I. Namely, let us rewrite $H=H_0+V$ as $H=H_F+H_I$, where

$$H_F=\int\bar\psi(\mathbf{x})[-i\boldsymbol{\gamma}\boldsymbol{\nabla}+m]\psi(\mathbf{x})d\mathbf{x}+\frac{1}{2}\int\ \left[\pi^2(\mathbf{x})+(\boldsymbol{\nabla}\phi(\mathbf{x}))^2+\mu^2\phi^2(\mathbf{x})\right]d\mathbf{x}, \qquad (2.6)$$

$$H_I=V+(m_0-m)\int\bar\psi(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}+\frac{1}{2}({\mu_0}^2-\mu^2)\int\ \phi^2(\mathbf{x})d\mathbf{x}\equiv V+M_{\rm ren}\ . \qquad (2.7)$$

The decomposition $H=H_F+H_I$ is the well-known trick (see, e.g., [15]), but this is not necessary for our clothing program: all the following results can be

obtained without the introduction of the mass counterterms $M_{\rm ren}$*. However, it simplifies the program realization. In other words, the separation $H = H_F + H_I$ may be justified not *ab initio* but *post factum*.

The operator H_F can be brought to the «diagonal» form**

$$H_F = \int \omega_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k}) d\mathbf{k} + \int E_{\mathbf{p}} \sum_r [b^\dagger(\mathbf{p},r) b(\mathbf{p},r) + d^\dagger(\mathbf{p},r) d(\mathbf{p},r)] d\mathbf{p} \quad (2.8)$$

by means of the standard expansions

$$\phi(\mathbf{x}) = (2\pi)^{-3/2} \int (2\omega_{\mathbf{k}})^{-1/2} [a(\mathbf{k}) + a^\dagger(-\mathbf{k})] \exp(i\mathbf{kx}) d\mathbf{k}\,, \quad (2.9)$$

$$\pi(\mathbf{x}) = -i(2\pi)^{-3/2} \int (\omega_{\mathbf{k}}/2)^{1/2} [a(\mathbf{k}) - a^\dagger(-\mathbf{k})] \exp(i\mathbf{kx}) d\mathbf{k}\,, \quad (2.10)$$

$$\psi(\mathbf{x}) = (2\pi)^{-3/2} \int (m/E_{\mathbf{p}})^{1/2} \sum_r [u(\mathbf{p},r) b(\mathbf{p},r) + v(-\mathbf{p},r) d^\dagger(-\mathbf{k},r)] \exp(i\mathbf{px}) d\mathbf{p}, \quad (2.11)$$

where $u(\mathbf{p},r)$ and $v(\mathbf{p},r)$ are the Dirac spinors, which satisfy the conventional equations $(\not{p} - m)u(\mathbf{p},r) = 0$ and $(\not{p} + m)v(\mathbf{p},r) = 0$ with $\not{p} = E_{\mathbf{p}}\gamma^0 - \mathbf{p}\boldsymbol{\gamma}$. In these formulae $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu^2}$.

2.3. The Unitary Transformation. The operators (2.3) are the corner-stone of the clothing procedure. Our aim is to find clothed operators which should satisfy the requirements i)–iv). Now, the symbol α will be used for set (2.3), with α_p being one operator of the set (cf. a and a_p). In order to implement the properties iii) and iv), we suppose that the clothed operators α are related to bare ones a via a unitary transformation

$$\alpha_p = W^\dagger a_p W, \quad W^\dagger W = W W^\dagger = 1\,, \quad (2.12)$$

where W is a function of all the bare operators a. Therefore, Eq. (2.12) represents α_p as a function (functional) of a.

Note that W is the same function of either clothed or bare operators (see [14]). Indeed, if $f(x)$ is a polynomial or a series of x, the relation $f(\alpha) = W^\dagger(a) f(a) W(a)$ follows from Eq. (2.12). Replacing $f(\alpha)$ by W leads to

$$W(\alpha) = W^\dagger(a) W(a) W(a) = W(a)\,, \quad (2.13)$$

*A simple example of the calculation of radiative correction to particle «bare» mass can be found in App. C.

**Nonessential c-number terms are henceforth omitted.

i.e., to the above statement. Hence, the operator a_p, when expressed in terms of α, is given by

$$a_p = W(\alpha)\ \alpha_p\ W^\dagger(\alpha)\ . \tag{2.14}$$

Unitarity of W is automatically ensured if W is represented as the exponential of an antihermitian operator R: $W = \exp R$. For a given R, the r.h.s. of Eq. (2.14) can be evaluated with the help of

$$\mathrm{e}^A B \mathrm{e}^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + ... \tag{2.15}$$

and the commutation rules (2.5).

In the context, the total Hamiltonian can be written as $H = H(a) = H_F + H_I$, where $H_F(a)$ is determined by Eq. (2.8) and $H_I = V(a) + M_{\mathrm{ren}}(a)$ with *

$$V(a) = \frac{ig}{(2\pi)^{3/2}} \int d\mathbf{p}'\ d\mathbf{p}\ d\mathbf{k}\ \frac{m}{(2\omega_{\mathbf{k}} E_{\mathbf{p}'} E_{\mathbf{p}})^{1/2}} \delta(\boldsymbol{p} + \boldsymbol{k} - \boldsymbol{p'}) \times$$

$$\times \{ \bar{u}(\boldsymbol{p'}r')\gamma_5 u(\boldsymbol{p}r)\ b^\dagger(\boldsymbol{p'}r') b(\boldsymbol{p}r)\ +$$

$$+\ \bar{u}(\boldsymbol{p'}r')\gamma_5 v(-\boldsymbol{p}r)\ b^\dagger(\boldsymbol{p'}r') d^\dagger(-\boldsymbol{p}r)\ +\ \bar{v}(-\boldsymbol{p'}r')\gamma_5 u(\boldsymbol{p}r)\ d(-\boldsymbol{p'}r') b(\boldsymbol{p}r)\ +$$

$$+\ \bar{v}(-\boldsymbol{p'}r')\gamma_5 v(-\boldsymbol{p}r)\ d(-\boldsymbol{p'}r') d^\dagger(-\boldsymbol{p}r) \} \ [a(\boldsymbol{k}) + a^\dagger(-\boldsymbol{k})]\ . \tag{2.16}$$

By using Eq. (2.14), one can replace the bare operators by the clothed ones

$$H(a) = H(W(\alpha)\alpha W^\dagger(\alpha)) \equiv K(\alpha)\ . \tag{2.17}$$

The operator $K(\alpha)$ represents the same Hamiltonian, but it has another dependence on its argument α compared to $H(a)$. $K(\alpha)$ can be found as follows. First, Eq. (2.17) can be written as

$$K(\alpha) = W(\alpha) H(\alpha) W^\dagger(\alpha)\ . \tag{2.18}$$

Second, putting $W(\alpha) = \exp R(\alpha)$ and using Eq. (2.15) we have

$$H = K(\alpha) = \mathrm{e}^R\ [H_F + H_I]\ \mathrm{e}^{-R} =$$

$$= H_F(\alpha) + H_I(\alpha) + [R, H_F] + [R, H_I] + \frac{1}{2}[R,[R,H_F]] + \frac{1}{2}[R,[R,H_I]] + ... \tag{2.19}$$

Eq. (2.19) gives a practical recipe for the $K(\alpha)$ calculation: at the beginning one replaces a by α in the initial expression $H(a)$ and then calculates $W(\alpha)H(\alpha)W^\dagger(\alpha)$ using Eqs. (2.15) and (2.5). The above transition $H(a) \to H(\alpha)$ generates a new operator $H(\alpha)$ as compared to $H(a)$, but Eqs. (2.17) and (2.18) show

*In cumbersome formulae summations over the dummy spin indices are sometimes omitted.

that $W(\alpha)H(\alpha)W^\dagger(\alpha)$ turns out to be equal to the original total Hamiltonian (cf. [35]).

We would like to stress that the transformation $WHW^\dagger$ under consideration should not be understood here as $W(a)H(a)W^\dagger(a)$. The latter would be a new operator $H'(a)$, which, in general, does not coincide with H. For a detailed discussion of different unitary transformations, see Sec. 6.

2.4. Elimination of «Bad» Terms. The next step is to fulfil the requirements i) – ii). If we want the no-clothed-particle state Ω and clothed one-particle states to be H eigenvectors, the r.h.s. of Eq. (2.19) must not contain some undesirable terms. Particularly, $K(\alpha)$ must not contain the $b_{\rm c}{}^\dagger d_{\rm c}{}^\dagger a_{\rm c}{}^\dagger$–type terms because they would give rise to the $f\bar{f}m$ states*, when acting on Ω, and $K(\alpha)\Omega$ could not be proportional to Ω. Similarly, $b_{\rm c}{}^\dagger b_{\rm c} a_{\rm c}{}^\dagger$ converts a one-fermion state $b_{\rm c}{}^\dagger\Omega$ into a fm state. But just terms of this kind enter into the operator $V(\alpha)$ which occurs in the r.h.s. of Eq. (2.19). In this connection, recall that $H_I(\alpha) = V(\alpha) + M_{ren}(\alpha)$, where $V(\alpha)$ is derived from $V(a)$, see Eq. (2.16), by means of the replacement $a \to \alpha$.

As we have argued above, the terms

$$b_{\rm c}{}^\dagger b_{\rm c} a_{\rm c}{}^\dagger,\ b_{\rm c}{}^\dagger d_{\rm c}{}^\dagger a_{\rm c},\ b_{\rm c}{}^\dagger d_{\rm c}{}^\dagger a_{\rm c}{}^\dagger,\ d_{\rm c} d_{\rm c}{}^\dagger a_{\rm c}{}^\dagger \tag{2.20}$$

in $V(\alpha)$ do not allow the clothed no-particle and one-particle states to be H eigenvectors. The remaining terms in $V(\alpha)$ are Hermitian conjugate of (2.20). We shall call «bad» all these terms. The contribution $b_{\rm c}{}^\dagger d_{\rm c}{}^\dagger a_{\rm c}{}^\dagger$ will be called the «bad» term of the class [3.0]: it is a product of three creation operators with destruction operators not included. The other three terms in (2.20) belong to the class [2.1]: two creation operators and one destruction operator.

The interaction H_I includes also the mass counterterm $M_{\rm ren}$ (see Eq. (2.7)). The latter contains bad terms of the class [2.0] (see, e.g., Eq. (2.27)). The self-energy correction to the particle mass can be represented by a series which starts with the terms of the g^2-order (see Eq. (2.31)). So, $M_{\rm ren}(\alpha) \sim g^2$ while $V(\alpha) \sim g^1$.

Let us eliminate from $K(\alpha)$ the bad terms of the g^1-order. For this purpose we choose such R that

$$V + [R, H_F] = 0\ . \tag{2.21}$$

One readily verifies that Eq. (2.21) cannot be satisfied until $R(\alpha)$ is linear or bilinear in α. To meet the equation, $R(\alpha)$ must be a three-operator, e.g., have the structure of $V(\alpha)$. Then, the commutator $[R, H_F]$ will also be three-operator expression since $H_F(\alpha)$ is the two-operator*.

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*With a transparent abbreviation $f\bar{f}m$ for a «fermion-antifermion-meson».

*Note that the commutator of the m-operator term and n-operator one yields a $(m+n-2)$-operator contribution.

Let us assume that the antihermitian $R(\alpha)$ contains the bad terms of the same kind as $V(\alpha)$ is. Namely, we put $R(\alpha) = \mathcal{R} - \mathcal{R}^\dagger$, where (cf. Eq. (2.16))

$$\mathcal{R} = \int d\mathbf{p}'\ d\mathbf{p}\ d\mathbf{k}\ \sum_{r'r} \{R_{11}^{\mathbf{k}}(\boldsymbol{p}'r';\boldsymbol{p}r) b_{\mathrm{c}}{}^\dagger(\boldsymbol{p}',r') b_{\mathrm{c}}(\boldsymbol{p},r) +$$

$$+ R_{12}^{\mathbf{k}}(\boldsymbol{p}'r';\boldsymbol{p}r) b_{\mathrm{c}}{}^\dagger(\boldsymbol{p}',r') d_{\mathrm{c}}{}^\dagger(-\boldsymbol{p},r) + R_{21}^{\boldsymbol{k}}(\boldsymbol{p}'r';\boldsymbol{p}r) d_{\mathrm{c}}(-\boldsymbol{p}',r') b_{\mathrm{c}}(\boldsymbol{p},r) +$$
$$+ R_{22}^{\mathbf{k}}(\boldsymbol{p}'r';\boldsymbol{p}r) d_{\mathrm{c}}(-\boldsymbol{p}',r') d_{\mathrm{c}}{}^\dagger(-\boldsymbol{p},r)\}\ a_{\mathrm{c}}(\boldsymbol{k})\ . \tag{2.22}$$

The c-number coefficients $R_{ij}^{\mathbf{k}}(i,j=1,2)$ are to be derived from Eq. (2.21), see Appendix A. We find that the solution exists if $\mu < 2m$. This condition has a clear physical meaning, viz., the meson can decay into the $\mathrm{f}\bar{\mathrm{f}}$-pair if $\mu > 2m$, and, therefore, one-meson state cannot be stable, i.e., it cannot be an H eigenvector. Once $[R, H_F] = -V$, Eq. (2.19) can be rewritten as

$$K(\alpha) = H_F(\alpha) + M_{\mathrm{ren}}(\alpha) + \frac{1}{2}[R,V] + [R, M_{\mathrm{ren}}] + \frac{1}{3}[R,[R,V]] + \ \ldots\ . \tag{2.23}$$

Thus we have removed from $K(\alpha)$ all the bad terms of the g^1-order.

However, the r.h.s. of Eq. (2.23) embodies other bad terms of the g^2- and higher orders. For example, $[R,V]$ contains the terms $g^2 b_{\mathrm{c}}{}^\dagger d_{\mathrm{c}}{}^\dagger a_{\mathrm{c}}{}^\dagger a_{\mathrm{c}}{}^\dagger$ of the class [4.0], which do not destroy the physical vacuum Ω (our evaluation of $[R,V]$ is given in Appendix A). In addition, we find in $[R,V]$ the terms $g^2 b_{\mathrm{c}}{}^\dagger d_{\mathrm{c}}{}^\dagger a_{\mathrm{c}}{}^\dagger a_{\mathrm{c}}$ of the class [3.1], which neither destroy $a_{\mathrm{c}}{}^\dagger\Omega$, nor retain it with a multiplicative factor. These and similar bad terms can be eliminated in a way analogous to the described above via one more transformation

$$\alpha_p = W_4(\alpha')\alpha_p' W_4{}^\dagger(\alpha')\ , \tag{2.24}$$

where $W_4 = \exp R_4$ and R_4 is an expression of the g^2-order, which consists of the above bad four-operator terms. This R_4 should be such that $[R_4, H_F]$ would cancel the latter terms.

Note also terms of the classes [2.0] and [1.1], which are present in M (see Eq. (2.27)), and similar terms, which appear after normal ordering of $[R,V]$ (see Appendix A). The bad terms of the class [2.0] must be removed from $K(\alpha)$ as well. We shall show in Subsec. 2.5 how they may be cancelled under a condition that relates the physical masses with the input parameters m_0, μ_0 and g.

Further, the double commutator $[R,[R,V]]$ in Eq. (2.23) is composed of five-operator terms (cf. the footnote on page 41), and there are bad terms among them. In particular, after reshuffling the operators into normal order, new three-operator bad terms occur. However, they are of the g^3-order. This type of bad terms can also be found in $[R,M]$. The subsequent unitary transformation makes it possible to remove all bad terms of the g^3-order.

Along the guideline, one may eliminate from the Hamiltonian the bad terms of increasing orders in the coupling constant g. It is assumed that in the limit the requirements i) and ii), which are equivalent to the absence of bad terms in K, will be fulfilled.

Finally, if our clothing procedure were perfect, the resulting representation K of the total Hamiltonian would possess the property

$$K(\alpha)|\ \mathbf{k}\rangle_c = H_F(\alpha)|\ \mathbf{k}\rangle_c = \omega_k|\ \mathbf{k}\rangle_c \tag{2.25}$$

with $|\ \mathbf{k}\rangle_c = {a_c}^\dagger(\mathbf{k})\Omega$. In other words, the new interaction term $K_I(\alpha) = K(\alpha) - H_F(\alpha)$ would satisfy the equation $K_I(\alpha)|\ \mathbf{k}\rangle_c = 0$. Analogous equations will hold for the physical vacuum Ω and the clothed one-fermion and one-antifermion states.

2.5. Particle Mass Renormalization. The cancellation of the bad two-operator terms in $K(\alpha)$ will be demonstrated for those of them which are bilinear in meson operators a_c and ${a_c}^\dagger$. The latter originate, first of all, from the meson mass counterterm

$$M_{\rm mes} = \frac{1}{2}({\mu_0}^2 - \mu^2)\int\ \phi^2(\mathbf{x})d\mathbf{x}\ . \tag{2.26}$$

Indeed, substituting the expansion (2.9) for $\phi(\mathbf{x})$ into Eq. (2.26), we obtain

$$M_{\rm mes} = \int\ \frac{{\mu_0}^2 - \mu^2}{4\omega_{\mathbf{k}}}[2{a_c}^\dagger(\mathbf{k})a_c(\mathbf{k}) + a_c(\mathbf{k})a_c(-\mathbf{k}) + {a_c}^\dagger(\mathbf{k}){a_c}^\dagger(-\mathbf{k})]d\mathbf{k}\ . \tag{2.27}$$

As shown in Appendix A, terms of the same operator structure occur in $K(\alpha)$ after normal ordering of the commutator $[R, V]$ from Eq. (2.23):

$$\int\ \frac{t_{\mathbf{k}}}{\omega_{\mathbf{k}}}[2{a_c}^\dagger(\mathbf{k})a_c(\mathbf{k}) + a_c(\mathbf{k})a_c(-\mathbf{k}) + {a_c}^\dagger(\mathbf{k}){a_c}^\dagger(-\mathbf{k})]d\mathbf{k}\ , \tag{2.28}$$

where $t_{\mathbf{k}}$ is determined by Eq. (A.20).

Now, we see that the sum of (2.27) and (2.28) gets equal to zero if

$${\mu_0}^2 - \mu^2 = -4t_{\mathbf{k}} \quad or \quad \mu^2 = {\mu_0}^2 + 4t_{\mathbf{k}}\ . \tag{2.29}$$

Here the quantity $t_{\mathbf{k}}$ must not depend on $\mathbf{k}$ along with μ^2 (cf. our argumentation in Subsec. 2.2). Unfortunately, this independence is not automatically provided in the Schroedinger picture that we use throughout. The integral that determines $t_{\mathbf{k}}$ (see Eq. (A.20)) is quadratically divergent and one needs to overcome the trouble (e.g., by introducing a cutoff factor). So, special efforts are required to yield the proof of independence (see, e.g., [5], Ch.6).

In the same way, the fermion mass counterterm

$$M_{\rm ferm} = (m_0 - m)\int \bar{\psi}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x} \tag{2.30}$$

cancels, under a proper condition, all the terms bilinear in the fermion operators, which arise from $\frac{1}{2}[R, V]$ as a result of normal ordering.

Up to now we have considered the bilinear terms of the g^2-order. Normal ordering of the six-operator and other terms of $K(\alpha)$ gives bad two-operator terms of the g^4- and higher orders. To eliminate them we suppose that $\delta\mu^2 \equiv {\mu_0}^2 - \mu^2$ and $\delta m \equiv m_0 - m$ may be expanded in the series

$$\delta\mu^2 = \sum_{n=1}^{\infty} g^{2n}(\delta\mu^2)_{2n}, \quad \delta m = \sum_{n=1}^{\infty} g^{2n}(\delta m)_{2n}. \tag{2.31}$$

Let us assume that $(\delta\mu^2)_2$ and $(\delta m)_2$ are used to remove the two-operator terms of the g^2-order as described above. Then, the terms $(\delta\mu^2)_4$ and $(\delta m)_4$ are destined to cancel two-operator terms of the g^4-order, and so on.

2.6. Some Remarks. So, the transformation realized by $W = \exp R$ fulfills an incomplete clothing, viz., it removes bad terms of the least order in g. The no-particle Ω and one-particle states ${b_c}^\dagger\Omega$, ${d_c}^\dagger\Omega$ and ${a_c}^\dagger\Omega$ constructed at this stage are merely approximate H eigenvectors.

We shall confine ourselves to the consideration of this transformation only while discussing the bound state problem like the deuteron (Sec. 4). Even this simplest application of our approach turns out to be rather cumbersome.

This section has been aimed to show how the unitary transformations of the original Hamiltonian for a system of interacting fields can be regarded as the introduction of new creation (destruction) operators instead of the initial «bare» ones. These new operators and corresponding one-particle states occurring at the first stage of the clothing procedure may be called «partially clothed». We juxtapose the one-particle states to the observable particles (say, pion and nucleon).

Let us note that the clothed states (operators) are not the in(out)-states (operators) of RQFT (see, e.g., [15], Ch. 17). Indeed, the two-particle in-states ${a_{in}}^\dagger(\mathbf{k}_1){a_{in}}^\dagger(\mathbf{k}_2)\Omega$ are H eigenstates while the two-particle clothed states ${a_c}^\dagger(\mathbf{k}_1){a_c}^\dagger(\mathbf{k}_2)\Omega$ do not (simple models with noninteracting particles are the evident exception). Within the in(out)-formalism it is *supposed* that H has the following eigenstates, viz., no-particle state (physical vacuum), states with one in-particle, two-, three-, etc., these states being analogous to the corresponding H_0 eigenstates. Meanwhile, the clothing formalism needs not such a supposition: one can find explicit expression for no- and one-particle clothed states in terms of bare states (using formulae of this section under the condition $\mu < 2m$). In addition, the in(out)-formalism does not consider two-particle states which have no H_0 bare partners, e.g., an in-state describing the deuteron. In the clothing formalism the problem of deuteron-like states is subject to further investigation (see Sec. 4).

3. GENERATORS FOR SPACE TRANSLATIONS AND SPACE-TIME ROTATIONS WITHIN THE CLOTHING PROCEDURE

In the previous section we have expressed the total Hamiltonian H in terms of the clothed operators. H is the time translation generator of the Poincaré group. Here we shall discuss how the rest of the group generators (total linear and angular momenta, and generators of Lorentz boosts) depend upon the clothed operators. This will allow us to formulate the transformation properties of the clothed no-particle and one-particle states under the Lorentz boosts. First of all, let us consider various constraints imposed on clothing transformations by the general symmetries in RQFT and those which are specific for a given field model.

3.1. Total Momenta and Other Motion Integrals in Terms of Clothed Operators. Similar to the determination of the total Hamiltonian H as a function $K(\alpha)$ of the clothed operators α we can obtain the expression $\boldsymbol{P}_{\mathbf{c}}(\alpha)$ for the total linear momentum $\boldsymbol{P}$ (cf. Eq. (2.17)):

$$\boldsymbol{P}(a) = \boldsymbol{P}(W(\alpha)\alpha W^{\dagger}(\alpha)) = W(\alpha)\boldsymbol{P}(\alpha)W^{\dagger}(\alpha) \equiv \boldsymbol{P}_{\mathbf{c}}(\alpha)\ . \qquad (3.1)$$

One can show that $W\boldsymbol{P}W^{\dagger}$ is simply equal to $\boldsymbol{P}$. Indeed, this is equivalent to $W\boldsymbol{P} = \boldsymbol{P}W$ that follows, in its turn, from $[R, \boldsymbol{P}] = 0$ (remind that $W = \exp R$). The validity of the latter can be verified using Eq. (2.22), (A.1), (A.2) and the well-known expression of the total linear momentum in terms of the creation (destruction) operators

$$\boldsymbol{P} = \int \mathbf{k} a^{\dagger}(\mathbf{k}) a(\mathbf{k}) d\mathbf{k} + \int \mathbf{p} \sum_{r} [b^{\dagger}(\mathbf{p}, r) b(\mathbf{p}, r) + d^{\dagger}(\mathbf{p}, r) d(\mathbf{p}, r)] d\mathbf{p} \qquad (3.2)$$

(see, e.g., Eq. (7.33) in [15]). Alternatively, one can take the representation (A.4) for R and use the equation $[\boldsymbol{P}, V] = 0$ that holds because of the invariance of V with respect to space translations.

So, we have found that $\boldsymbol{P}_{\mathbf{c}}(\alpha) = \boldsymbol{P}(\alpha)$. This means that $\boldsymbol{P}$ is the same function of the clothed operators as of the bare ones.

Analogous statements are valid for the total angular momentum $\boldsymbol{M}$* and the baryon (fermion) number operator $B = \int \psi^{\dagger}(\mathbf{x})\psi(\mathbf{x})d\mathbf{x}$. This means that the clothed states Ω, ${a_{\mathrm{c}}}^{\dagger}\Omega$, ${b_{\mathrm{c}}}^{\dagger}\Omega$, and ${d_{\mathrm{c}}}^{\dagger}\Omega$ have the following properties: a) they are eigenvectors of $\boldsymbol{P}$; b) they are transformed under space rotations in the same manner as the relevant bare states do; c) they possess definite B values.

Instead of verifying the properties $[R, \boldsymbol{P}] = [R, \boldsymbol{M}] = [R, B] = 0$ with the solution R of the equation $[R, H_F] + V = 0$ one may consider them as

*The field-theoretical formula for the generator can be found in [38](Ch.11), see also Subsec. 3.2.

some new requirements supplementary to those listed in Subsec. 2.1 as i) – iv). These requirements would result in definite restrictions on the coefficients involved in the expansion (2.22) for R. For example, it follows from $[R, \boldsymbol{P}] = 0$ that $R^{\mathbf{k}}_{ij}(\boldsymbol{p'}r'; \boldsymbol{p}r)$ must have the form $\delta(\mathbf{p} + \mathbf{k} - \mathbf{p}')r^{\mathbf{k}}_{ij}(\boldsymbol{p'}r'; \boldsymbol{p}r)$. The equation $[R, \boldsymbol{M}] = 0$ means that $R^{\mathbf{k}}_{ij}(\boldsymbol{p'}r'; \boldsymbol{p}r)$ must depend on rotationally invariant combinations of its arguments. Besides, the condition $[R, B] = 0$ prevents R to be dependent on terms of the ${b_{\mathrm{c}}}^{\dagger}d_{\mathrm{c}}$- kind (we eliminate such terms from the beginning assuming the form (2.22) for R).

One may add to these restrictions those which are consequences of the following requirements: clothed operators and clothed one-particle states must have the same transformation properties with respect to space inversion, time reversal and charge conjugation as their bare partners.

Let us stress that all the constraints are exact whereas the equation $[R, H_F] + V = 0$ considered in Subsec. 2.4 is merely approximate one. However, its solution has all the properties in question since the interaction V commutes with $\boldsymbol{P}$, $\boldsymbol{M}$, B, etc. (see Eq. (A.4)).

3.2. Transformations of Bare Operators and States under Lorentz Boosts. A distinctive feature of the conventional relativistic dynamics («instant» form after Dirac [39]) is that the generators $\boldsymbol{N} = (N^1, N^2, N^3)$ of the Lorentz boost $\Lambda = \exp(i\boldsymbol{\beta}\boldsymbol{N})^{**}$ contain interaction terms while the linear $\mathbf{P} = (P^1, P^2, P^3)$ and angular $\mathbf{M} = (M^1, M^2, M^3)$ momenta are determined by the same expressions as for free fields.

In order to see this explicitly let us resort to the Lagrangian formalism where the quantities of fundamental importance are the energy-momentum density tensor $\mathcal{T}^{\mu\nu}(x)$ and the angular momentum density tensor $\mathcal{M}^{\lambda\mu\nu}(x)^{***}$ (see, e.g., Ch.11 in [38]). By definition, $P^{\mu} = (H, \mathbf{P}) = \int \mathcal{T}^{0\mu}(x)d\mathbf{x}$, $M^j = \epsilon_{jkl}M^{kl}$, and $N^j = M^{0j}$, where $M^{\mu\nu} = \int \mathcal{M}^{0\mu\nu}(x)d\mathbf{x}$. According to the Noether theorem all the ten operators are time independent, i.e., they are the motion integrals. In other words, they can be evaluated at $t = 0$, i.e., they can be expressed through field operators in the Schrödinger picture.

The corresponding representation of $\mathbf{N}$ depends on the form of $\mathcal{T}^{\mu\nu}(x)$ (nonsymmetrized or symmetrized*), that is utilized. Here we shall use the nonsym-

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**Here $\boldsymbol{\beta} = \beta\mathbf{n}$, $\mathbf{n} = \frac{\boldsymbol{v}}{v}$ and $th\beta = v$, where v is the velocity of a reference frame moving along the $\mathbf{n}$ direction. In this paper we use the system of units in which the light velocity c is equal to unity.

***Greek labels run the values 0,1,2,3.

*The symmetrized form with the Belinfante ansatz [36] for Lorentz boosts has been employed in our talks [19, 20]. Another application of the form can be found in a covariant description of electromagnetic interactions with nuclei [40].

metrized form (see Eqs. (13.45) and (13.47) in [38]), which leads to

$$\mathbf{N} = \mathbf{N}_F - \int \mathbf{x} V(\mathbf{x}) d\mathbf{x} + \mathbf{N}_{\rm ren} \ , \tag{3.3}$$

where $\mathbf{N}_F$ is the free part of $\mathbf{N}$:

$$\mathbf{N}_F = \mathbf{N}_{\rm ferm} + \mathbf{N}_{\rm mes} \ , \tag{3.4}$$

$$\mathbf{N}_{\rm ferm} = -\int \mathbf{x} \bar{\psi}(\mathbf{x}) [-i\boldsymbol{\gamma}\boldsymbol{\nabla} + m] \psi(\mathbf{x}) d\mathbf{x} + \frac{i}{2} \int \bar{\psi}(\mathbf{x}) \boldsymbol{\gamma} \psi(\mathbf{x}) d\mathbf{x} \, , \tag{3.5}$$

$$\mathbf{N}_{\rm mes} = -\frac{1}{2} \int \mathbf{x} \left[\pi^2(\mathbf{x}) + (\boldsymbol{\nabla}\phi(\mathbf{x}))^2 + \mu^2 \phi^2(\mathbf{x}) \right] d\mathbf{x} \ . \tag{3.6}$$

In accordance with the particle mass renormalization described in Subsec. 2.5 we have separated the contribution to $\mathbf{N}$ from meson and fermion mass counterterms (cf. Eqs. (2.26) and (2.30)):

$$\mathbf{N}_{\rm ren} = \mathbf{N}_{\rm ren}^{\rm mes} + \mathbf{N}_{\rm ren}^{\rm ferm} \, , \tag{3.7}$$

$$\mathbf{N}_{\rm ren}^{\rm mes} = -\frac{1}{2} ({\mu_0}^2 - \mu^2) \int \mathbf{x} \phi^2(\mathbf{x}) d\mathbf{x} \, , \tag{3.8}$$

$$\mathbf{N}_{\rm ren}^{\rm ferm} = -(m_0 - m) \int \mathbf{x} \bar{\psi}(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x} \ . \tag{3.9}$$

Now, to express the generators through the creation (destruction) operators let us take the expansions (2.9)–(2.11) and employ the relation

$$\int x^j \exp{(i\boldsymbol{q}\mathbf{x})} d\mathbf{x} = -i(2\pi)^3 \frac{\partial}{\partial q^j} \delta(\boldsymbol{q}).$$

Then, e.g., we find

$$N^j_{\rm mes} = \frac{i}{2} \int a^\dagger(\mathbf{k}') a(\mathbf{k}) \frac{\omega_{\mathbf{k}'} \omega_{\mathbf{k}} + \mathbf{k}'\mathbf{k} + \mu^2}{\sqrt{\omega_{\mathbf{k}'} \omega_{\mathbf{k}}}} \frac{\partial}{\partial k^j} \delta(\mathbf{k}' - \mathbf{k}) d\mathbf{k}' d\mathbf{k} \tag{3.10}$$

or

$$N^j_{\rm mes} = \frac{i}{2} \int \omega_{\mathbf{k}} \big[\frac{\partial a^\dagger(\mathbf{k})}{\partial k^j} a(\mathbf{k}) - a^\dagger(\mathbf{k}) \frac{\partial a(\mathbf{k})}{\partial k^j} \big] d\mathbf{k} \ . \tag{3.10$'$}$$

Simultaneously, we get (cf. Eq. (2.16))

$$N^j_I \equiv -\int x^j V(\mathbf{x}) d\mathbf{x} =$$

$$= \frac{ig}{(2\pi)^{3/2}} \int d\mathbf{p}' \ d\mathbf{p} \ d\mathbf{k} \ \frac{m}{(2\omega_{\mathbf{k}} E_{\mathbf{p}'} E_{\mathbf{p}})^{1/2}} \ \frac{\partial}{\partial k^j} \delta(\boldsymbol{p} + \boldsymbol{k} - \boldsymbol{p'}) \times$$

$$\times\{\bar{u}(\boldsymbol{p}'r')\gamma_5 u(\boldsymbol{p}r)\ b^\dagger(\boldsymbol{p}'r')b(\boldsymbol{p}r)\ +$$
$$+\ \bar{u}(\boldsymbol{p}'r')\gamma_5 v(-\boldsymbol{p}r)\ b^\dagger(\boldsymbol{p}'r')d^\dagger(-\boldsymbol{p}r)\ +\ \bar{v}(-\boldsymbol{p}'r')\gamma_5 u(\boldsymbol{p}r)\ d(-\boldsymbol{p}'r')b(\boldsymbol{p}r)\ -$$
$$-\ \bar{v}(-\boldsymbol{p}'r')\gamma_5 v(-\boldsymbol{p}r)\ d^\dagger(-\boldsymbol{p}r)d(-\boldsymbol{p}'r')\}\ [a(\boldsymbol{k})+a^\dagger(-\boldsymbol{k})] \tag{3.11}$$

with the Yukawa interaction density $V(\mathbf{x})$ (see Eq. (2.2)).

These formulae enable one to perform directly transformations of the bare operators and states under the Lorentz boosts. In particular, in the infinitesimal case with $\mid \beta^j \mid \ll 1$ (j = 1,2,3) one has

$$\Lambda\Omega_0 = \exp\left[i\boldsymbol{\beta}(\mathbf{N}_F+\mathbf{N}_I+\mathbf{N}_{\rm ren})\right]\Omega_0 \simeq$$
$$\simeq [1+i\boldsymbol{\beta}(\mathbf{N}_I+\mathbf{N}_{\rm ren})]\Omega_0 \simeq (1+i\boldsymbol{\beta}\mathbf{N}_I)\Omega_0\ , \tag{3.12}$$

where we took advantage of the relation

$$\mathbf{N}_F\Omega_0 = 0\ , \tag{3.13}$$

and omit terms of the order higher than g^1.

Equation (3.12) means that the bare vacuum is not invariant with respect to Λ, viz., $\Lambda\Omega_0 \neq \Omega_0$. A moving observer «sees» $\Lambda\Omega_0$ as the superposition of no-particle state Ω_0, $N\bar{N}\pi$ states, etc.

Similarly, bare one-particle states (e.g., $a^\dagger(k)\Omega_0$) are not transformed with respect to Λ as in the free case where along with Eq. (3.13) one has the property*

$$\mathrm{e}^{i\boldsymbol{\beta}\mathbf{N}_F}a^\dagger(k)\mathrm{e}^{-i\boldsymbol{\beta}\mathbf{N}_F} = a^\dagger(Lk)\ . \tag{3.14}$$

In Eq. (3.14) L denotes a pure Lorentz transformation («boost») with the matrix:

$$L = \left[L^\mu_\nu\right] = \begin{bmatrix} & u^0 & \vdots & u_j & \\ & \cdots & \cdots & \cdots & \\ -u^i & & \vdots & \delta^i_j - \frac{u^i u_j}{1+u_0} & \end{bmatrix}\ , \tag{3.15}$$

where $u^\mu = (u^0, \boldsymbol{u}) = (ch\beta, \mathbf{n}sh\beta)$ is the four-velocity vector. The boost converts the four-momentum $k = (\omega_{\mathbf{k}}, \mathbf{k})$ into $k' = Lk = (\omega_{\mathbf{k}'}, \mathbf{k}')$.

We have $\Lambda a^\dagger(k)\Omega_0 = {a_\Lambda}^\dagger(k)\Lambda\Omega_0$ with the transformed meson operator ${a_\Lambda}^\dagger(k) = \Lambda a^\dagger(k)\Lambda^\dagger$. For an infinitesimal boost the operator

$${a_\Lambda}^\dagger(k) = \mathrm{e}^{i\boldsymbol{\beta}\mathbf{N}}a^\dagger(k)\mathrm{e}^{-i\boldsymbol{\beta}\mathbf{N}} \simeq a^\dagger(k) + i\boldsymbol{\beta}[\mathbf{N}_{\rm mes}, a^\dagger(k)] + i\boldsymbol{\beta}[\mathbf{N}_I, a^\dagger(k)] \tag{3.16}$$

---

*Under the discussion it is convenient to proceed with the operators $a(k)$ which obey the covariant commutation relation $[a(k), a^\dagger(k')] = \omega_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}')$ (cf., e.g., Eq. (7.23) in [15]). In the framework of our consideration it is equivalent to replacement of $a(\mathbf{k})$ by $a(k)/\sqrt{\omega_{\mathbf{k}}}$. It is true for the respective clothed operators, so that, for instance, the first of the relations (2.5) should be replaced by $[a_{\rm c}(k), {a_{\rm c}}^\dagger(k')] = \omega_{\mathbf{k}}\delta(\mathbf{k}-\mathbf{k}')$.

contains bare fermion operators due to the last («interaction») term in the r.h.s. of the equation. By using Eq. (3.10′) one can see that the two first terms yield $a^\dagger(k) - \beta^j \frac{\partial a^\dagger(k)}{\partial k^j}\omega_{\mathbf{k}} \simeq a^\dagger(\omega_{\mathbf{k}} - \boldsymbol{\beta}\mathbf{k}, \mathbf{k} - \omega_{\mathbf{k}}\boldsymbol{\beta})$ that coincides with the operator $a^\dagger(Lk)$ in Eq. (3.14) for the infinitesimal L.

Now, taking into account Eqs. (3.12) and (3.16), one can ascertain that the transformed state $\Lambda a^\dagger(k)\Omega_0$ is composed of the one-meson state $a^\dagger(k')\Omega_0$ with the properly changed momentum $\mathbf{k}' = \mathbf{k} - \omega_{\mathbf{k}}\boldsymbol{\beta}$ and the states $\mid f\bar{f}\rangle$ and $\mid f\bar{f}\pi\pi\rangle$ containing fermions.

3.3. Boost Generators for Clothed Particles. Elimination of Bad Terms. It is reasonable to anticipate that the physical vacuum Ω and clothed one-particle states (e.g., ${a_c}^\dagger(k)\Omega$) should be, respectively, the no-clothed-particle state and clothed one-particle states from the point of view of a moving observer. More exactly, they should meet the relations

$$\Lambda\Omega = \Omega \tag{3.17}$$

and

$$\Lambda {a_c}^\dagger(k)\Omega = {a_c}^\dagger(Lk)\Omega\ . \tag{3.18}$$

The previous experience of handling with Eqs. (3.12) and (3.16) prompts that these conditions could be provided (at least, approximately) if we shall manage to remove bad terms from $\mathbf{N}$ (in practice, some of them) while expressing it through the clothed operators. In this connection, let us write down

$$\mathbf{N} \equiv \mathbf{N}(a) = W\mathbf{N}(\alpha)W^\dagger \equiv \mathbf{B}(\alpha) = \mathrm{e}^{R(\alpha)}[\mathbf{N}_F(\alpha) + \mathbf{N}_I(\alpha) + \mathbf{N}_{\mathrm{ren}}(\alpha)]\mathrm{e}^{-R(\alpha)} =$$

$$= \mathbf{N}_F + \mathbf{N}_I + [R, \mathbf{N}_F] + [R, \mathbf{N}_I] + ... \tag{3.19}$$

(cf. Eq. (2.19)) and then remove from the r.h.s. of (3.19) all the bad terms of the g^1-order by requiring that

$$[N_F^j, R] = N_I^j = -\int x^j V(\mathbf{x})d\mathbf{x} \quad (j = 1, 2, 3)\ . \tag{3.20}$$

Now, we want to show that Eqs. (3.20) will automatically hold if R satisfies the condition (2.21). For this purpose, one can use the representation (A.4) for such R:

$$R = -i \lim_{\epsilon \to 0^+} \int_0^\infty dt \mathrm{e}^{-\epsilon t} \int V(x)d\mathbf{x}\ , \tag{A.4}$$

where $V(x) = V(\mathbf{x}, t) = \mathrm{e}^{iH_F t}V(\mathbf{x})\mathrm{e}^{-iH_F t}$ is the interaction operator in the Dirac picture. Being a scalar, it is transformed as

$$\mathrm{e}^{i\boldsymbol{\beta}\mathbf{N}_F}V(x)\mathrm{e}^{-i\boldsymbol{\beta}\mathbf{N}_F} = V(Lx) \tag{3.21}$$

under the Lorentz boost L determined by the matrix (3.15). Note that one can directly verify the validity of Eq. (3.21) in case when N_F is the Schroedinger operator (as it does in Eq. (3.19)) while $V(x)$ being any Lorentz invariant operator in the Dirac picture.

In order to exploit this property note that

$$[N_F^1, R] = -i\frac{\partial}{\partial\beta^1}\left[\mathrm{e}^{i\beta^1 N_F^1} R \mathrm{e}^{-i\beta^1 N_F^1}\right]\Big|_{\beta^1=0} \tag{3.22}$$

or taking into account firstly Eq. (A.4) and then Eq. (3.21),

$$[N_F^1, R] = -\lim_{\epsilon\to 0^+}\lim_{\beta^1\to 0}\frac{\partial}{\partial\beta^1}\int_0^\infty dt\mathrm{e}^{-\epsilon t}\int V(Lx)d\mathbf{x}. \tag{3.23}$$

For the infinitesimal boost we have

$$V(Lx) = V(\mathbf{x} - \boldsymbol{\beta}t, t - \boldsymbol{\beta}\mathbf{x}) = V(x^1 - \beta^1 t, x^2, x^3, t - \beta^1 x^1) \tag{3.24}$$

and

$$\begin{aligned}[N_F^1, R] &= -\lim_{\epsilon\to 0^+}\int_0^\infty dt\mathrm{e}^{-\epsilon t}\int \lim_{\beta^1\to 0}\frac{\partial}{\partial\beta^1}V(x^1 - \beta^1 t, x^2, x^3, t - \beta^1 x^1)d\mathbf{x} = \\ &= -\lim_{\epsilon\to 0^+}\int_0^\infty dt\mathrm{e}^{-\epsilon t}\int(-t\frac{\partial}{\partial x^1}V(\mathbf{x},t) - x^1\frac{\partial}{\partial t}V(\mathbf{x},t))d\mathbf{x} = \\ &= \lim_{\epsilon\to 0^+}\Big[\int x^1 d\mathbf{x}\int_0^\infty \mathrm{e}^{-\epsilon t}\frac{\partial}{\partial t}V(\mathbf{x},t)dt + \int_0^\infty tdt\mathrm{e}^{-\epsilon t}\int\frac{\partial}{\partial x_1}V(\mathbf{x},t)d\mathbf{x}\Big]\ .\end{aligned} \tag{3.25}$$

One can show (cf., the proof of the relation (A.4)) that the first term in the square brackets yields $-\int x^1 V(\mathbf{x})d\mathbf{x}$. At the same time the second term is equal to zero since the operator $V(\mathbf{x},t)$ (more exactly, its matrix elements) vanishes at $x^1 = \pm\infty$. So, we get the desirable relation (3.20) with $j = 1$ (the cases $j = 2, 3$ are analogous).

So, the transformation $W = \exp R$ eliminates simultaneously the three-operator terms $\sim g^1$ both from the total Hamiltonian $K(\alpha)$ and from the boost generators $\mathbf{B}(\alpha)$. One should emphasize that the proof is valid for any Lorentz scalar function $V(\mathbf{x},t)$. Specific expressions for $\mathbf{N}_F$ have not been required as well since all we have needed is Eq. (3.21).

After this elimination of bad terms we get by analogy to Eq. (2.23),

$$\mathbf{B}(\alpha) = \mathbf{N}_F(\alpha) + \mathbf{N}_{\mathrm{ren}}(\alpha) + \frac{1}{2}[R, \mathbf{N}_I] + [R, \mathbf{N}_{\mathrm{ren}}] + \frac{1}{3}[R, [R, \mathbf{N}_I]] + ... \tag{3.26}$$

We shall not exemplify separate interaction terms in the r.h.s. of this equation since their structure repeats that for the corresponding contributions to $K(\alpha)$ (cf., e.g., Eqs. (3.11) and (2.16)).

4. EQUATIONS FOR BOUND AND SCATTERING STATES IN RQFT

The clothed one-particle states are eigenstates of the total Hamiltonian H according to their definitions (see Subsec. 2.1). There may be other H eigenstates which describe physical systems resembling one-particle states, viz., the states with discrete values of the system mass that may be defined as the system energy in the rest frame of reference (scattering states belong to continuous values of the mass).

First of all, we keep in mind the simplest bound states similar to the hydrogen atom or the deuteron. In the non–relativistic approach the wavefunction of such two-body state is the product of a function that describes the system as a whole and other function being dependent on the internal variables (for instance, the relative momentum $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$ for two identical particles). Therefore, the centre-of-mass motion is separated from the internal motion. It is not the case for any relativistic model which satisfies the Poincaré algebra. In fact, the coupling between the internal and centre-of-mass motions is inherent to relativistic theories of interacting particles (see, e.g., the papers [22,23] where within simple field models the two-body bound states are studied in a moving reference frame).

For the Yukawa model the corresponding states may be fermion–fermion states (deuteron-like), meson-fermion ones, etc. Of course, we are not able to find the exact Hamiltonian eigenstates, except some exactly solvable models (see [14]). However, within the clothing procedure in question one can suggest reasonable approximations to this problem.

4.1. New Zeroth Approximation for the Total Hamiltonian K. Our approach is based on the choice of an appropriate zeroth approximation (ZA) to the total Hamiltonian expressed through the clothed operators, i.e., the operator K determined by Eq. (2.17). Since its free part $K_2 = H_F(\alpha)$ has no deuteron-like eigenstates, we shall try to take

$$K_{ZA} = K_2 + g^2 K_4^{(2)} \tag{4.1}$$

by adding to the two-operator (one-body) contribution

$$K_2 = \int \omega_{\mathbf{k}} {a_c}^{\dagger}(\mathbf{k}) a_c(\mathbf{k}) d\mathbf{k} +$$

$$+ \int E_{\mathbf{p}} \sum_r [{b_c}^{\dagger}(\mathbf{p}, r) b_c(\mathbf{p}, r) + {d_c}^{\dagger}(\mathbf{p}, r) d_c(\mathbf{p}, r)] d\mathbf{p} \equiv K_\pi + K_N \tag{4.2}$$

the four-operator (two-body) contributions of the g^2-order which arise from the commutator $\frac{1}{2}[R, V] \equiv \frac{1}{2}[R_3, V]$ in the r.h.s. of Eq. (2.23). This commutator is evaluated in Appendix A. Doing so, we obtain the decomposition

$$K_4 \equiv g^2 K_4^{(2)} = K(NN \to NN) + K(\bar{N}\bar{N} \to \bar{N}\bar{N}) + K(N\bar{N} \to N\bar{N}) +$$

$$+K(\pi N \to \pi N) + K(\pi \bar{N} \to \pi \bar{N}) + K(\pi\pi \to N\bar{N}) + K(N\bar{N} \to \pi\pi) \quad (4.3)$$

with the separate interactions between the different clothed particles. They are displayed (very schematically) in Fig. 1 where the graph (*a*) represents the nucleon–nucleon interaction

$$K(NN \to NN) = \sum_{r,r'} \int d\mathbf{p}_1' d\mathbf{p}_2' d\mathbf{p}_1 d\mathbf{p}_2 \times$$

$$\times V_{NN}(\mathbf{p}_1', r_1', \mathbf{p}_2', r_2'; \mathbf{p}_1, r_1, \mathbf{p}_2, r_2) b_c{}^\dagger(\mathbf{p}_1', r_1') b_c{}^\dagger(\mathbf{p}_2', r_2') b_c(\mathbf{p}_1, r_1) b_c(\mathbf{p}_2, r_2) \quad (4.4)$$

while the pion–nucleon interaction

$$K(\pi N \to \pi N) = \sum_{r,r'} \int d\mathbf{k}' d\mathbf{p}' d\mathbf{k} d\mathbf{p} \times$$

$$\times V_{\pi N}(\mathbf{k}', \mathbf{p}', r'; \mathbf{k}, \mathbf{p}, r) a_c{}^\dagger(\mathbf{k}') b_c{}^\dagger(\mathbf{p}', r') a_c(\mathbf{k}) b_c(\mathbf{p}, r) \quad (4.5)$$

is displayed by the graph (*c*).

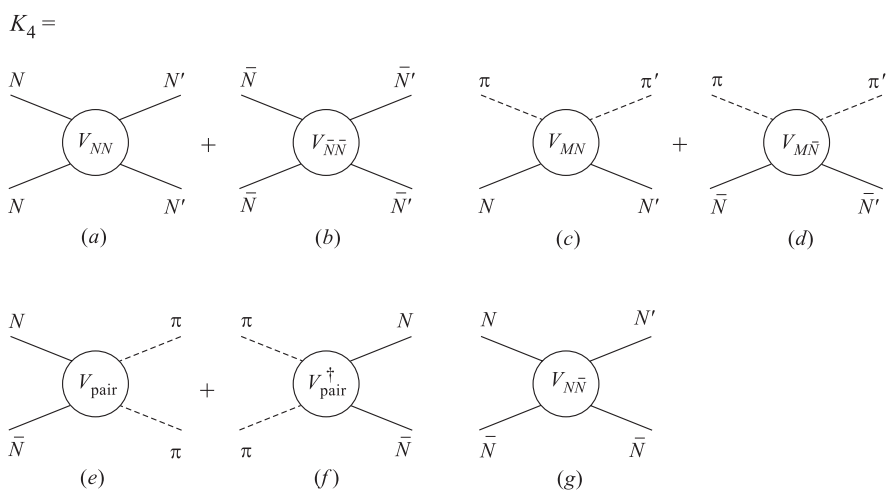

Fig. 1. Schematic representation of separate contributions to the effective operator K_4: *a)* $b_c{}^\dagger b_c{}^\dagger b_c b_c$, *b)* $d_c{}^\dagger d_c{}^\dagger d_c d_c$, *c)* $b_c{}^\dagger a_c{}^\dagger b_c a_c$, *d)* $d_c{}^\dagger a_c{}^\dagger d_c a_c$, *e)* $b_c{}^\dagger d_c{}^\dagger a_c a_c$, *f)* $a_c{}^\dagger a_c{}^\dagger d_c b_c$, *g)* $b_c{}^\dagger d_c{}^\dagger d_c b_c$

Explicit expressions for the coefficients $V_{\pi N}$ and V_{NN} will be given in the next Subsecs. Here, however, one should note that all these terms of $K_4^{(2)}$ describe only real processes such as $N + N \to N + N$, $\pi + N \to \pi + N$, etc.

The bad terms of $[R_3, V]$ are not included in K_{ZA} for the reasons discussed in Subsecs. 2.4 and 2.5. For example, the terms of the kind [4.0] (e.g., $b^\dagger d^\dagger b^\dagger d^\dagger$ and $b^\dagger d^\dagger a^\dagger a^\dagger$) and [3.1] (e.g., $b^\dagger d^\dagger b^\dagger b$ and $b^\dagger d^\dagger a^\dagger a$) must be removed during our clothing procedure via the transformation $W_4 = \exp R_4$ (see Eq. (2.24)). Recall that $[R_4, K_2]$ must cancel these bad terms.

In its turn, the transformation W_4 (see Eq. (2.23) for K)

$$W_4 K(\alpha') W_4{}^\dagger = W_4(K_2(\alpha') + \frac{1}{2}[R_3(\alpha'), V(\alpha')] + ...)W_4{}^\dagger =$$

$$= K_2(\alpha') + \frac{1}{2}[R_3(\alpha'), V(\alpha')]_4 + [R_4, K_2] + \frac{1}{2}[R_4, [R_3, V]_4] + ... \ , \qquad (4.6)$$

brings in the Hamiltonian new four-operator terms in addition to those mentioned above, see Eq. (4.3). Here, $[R_3, V]_4$ denotes the four-operator part of $[R_3, V]$*. After the normal ordering, the double commutator $[R_4, [R_3, V]_4]$ yields new four-operator interactions in the total Hamiltonian. However, they are of the g^4-order whereas the interaction terms which we have included in K_{ZA} are of the g^2-order. So, the latter are not altered by W_4. They exhaust all the interaction terms of the g^2-order, which remain in K_{ZA}.

We hope that K_{ZA} eigenstates are good approximations to exact K eigenstates. The former should be found nonperturbatively (e.g., by means of numerical methods). Five-operator and more complicated interaction terms can be taken into account via perturbation theory recipes.

4.2. Meson-Nucleon Eigenstates of K_{ZA} and Pion-Nucleon Quasipotential. The operator K_{ZA} has an important property: it conserves the total number of clothed particles. In particular, K_{ZA} transforms clothed two-particle states (e.g., of the NN or πN types) to two-particle ones. Moreover, the Fock subspace of all the clothed states can be divided into several sectors (the NN sector, the πN sector, etc.) such that K_{ZA} leaves each of them to be invariant, i.e., for any state vector Φ of such sector $K_{ZA}\Phi$ belongs to the same sector.

Let us show in the simplest case of the πN sector that the property just mentioned allows us to reduce the eigenstate equation $K_{ZA}\Phi^E = E\Phi^E$ to the related Schroedinger equation of the particle-number-conserving quantum mechanics. For this purpose we seek Φ^E as the following superposition of the πN sector states ${a_c}^\dagger {b_c}^\dagger \Omega$,

$$\Phi^E_{\pi N} = \sum_r \int d\mathbf{k} d\mathbf{p} \Phi^E_{\pi N}(\mathbf{k}; \mathbf{p}, r) {a_c}^\dagger(\mathbf{k}) {b_c}^\dagger(\mathbf{p}, r)\Omega \ . \qquad (4.7)$$

*The two-operator terms of $[R_3, V]$ are supposed to be cancelled with the respective mass counter terms.

In the πN sector K_{ZA} is equal to $K_\pi + K_N + K(\pi N \to \pi N)$ because the rest terms of K_{ZA} give zero when acting on $\Phi^E_{\pi N}$. Thus, the equation $K_{ZA}\Phi^E_{\pi N} = E\Phi^E_{\pi N}$ reduces to

$$[K_\pi + K_N + K(\pi N \to \pi N)]\Phi^E_{\pi N} = E\Phi^E_{\pi N} \ . \tag{4.8}$$

Taking the scalar products of both the parts of (4.8) with $\langle {a_c}^\dagger(\mathbf{k}){b_c}^\dagger(\mathbf{p}, r)\Omega \mid$, we get the relevant equation for $\Phi^E_{\pi N}(\mathbf{k}; \mathbf{p}, r)$,

$$(E - \omega_{\mathbf{k}} - E_{\mathbf{p}})\Phi^E_{\pi N}(\mathbf{k}; \mathbf{p}, r) = \sum_{r'} \int d\mathbf{k}' d\mathbf{p}' V_{\pi N}(\mathbf{k}, \mathbf{p}, r; \mathbf{k}', \mathbf{p}', r')\Phi^E_{\pi N}(\mathbf{k}'; \mathbf{p}', r') \ . \tag{4.9}$$

The kernel $V_{\pi N}$ of this integral equation is determined by Eq. (4.5) (an explicit expression for it is given below). This kernel can be called a quasi-potential: in the coordinate representation it may depend not only on the particle coordinates but on their derivatives as well. In our opinion, a popular name «effective Hamiltonian» is inappropriate for K_{ZA}. When one deals with an effective Hamiltonian one has to argue that its eigenvalues and eigenvectors coincide (at least, approximately) with those of an original Hamiltonian. In the framework of our approach we do not need such a proof since we believe that K_{ZA} is in a sense a major part of the total Hamiltonian H and, therefore, the above approximate coincidence is provided.

The solutions of Eq. (4.9), which belong to a discrete spectrum (if it exists), describe πN bound states. As emphasized above, the solutions should be found nonperturbatively. Continuous πN-mass values correspond to πN-scattering states. The respective S-matrix elements may be evaluated either exactly by using numerical methods of solving the Lippmann–Schwinger equation for the T-matrix with the interaction $K(\pi N \to \pi N)$ (see, e.g., [41]), or approximately in the framework of old–fashioned noncovariant perturbation theory (see, e.g., Ch.1 in [42] or Ch.11 in [15]).

To obtain the explicit expression for $K(\pi N \to \pi N)$, one needs to seperate out the ${a_c}^\dagger {b_c}^\dagger a_c b_c$-kind terms of the commutators $[\mathcal{R}, \mathcal{V}^\dagger]$ and $[\mathcal{R}, \mathcal{V}^\dagger]^\dagger$ (see Eq. (A.14))

$$K(\pi N \to \pi N) = \frac{1}{2} : \left\{ [\mathcal{R}, \mathcal{V}^\dagger]_{\pi N} + [\mathcal{R}, \mathcal{V}^\dagger]^\dagger_{\pi N} \right\} :=$$

$$= -\frac{1}{2} \int d\mathbf{k}_2 d\mathbf{p}_2 d\mathbf{k}_1 d\mathbf{p}_1 \times$$

$$\times \left\{ [V^{-\mathbf{k}_2}, R^{\mathbf{k}_1}]_{11}(\mathbf{p}_2 r_2; \mathbf{p}_1 r_1) + [V^{-\mathbf{k}_1}, R^{\mathbf{k}_2}]^\dagger_{11}(\mathbf{p}_2 r_2; \mathbf{p}_1 r_1) \right\} \times$$

$$\times {a_c}^\dagger(\mathbf{k}_2){b_c}^\dagger(\mathbf{p}_2, r_2) a_c(\mathbf{k}_1) b_c(\mathbf{p}_1, r_1) \ , \tag{4.10}$$

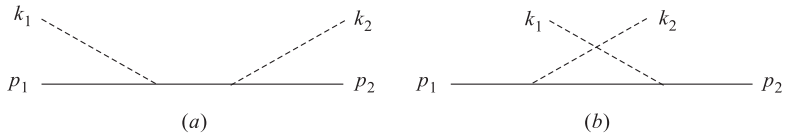

Fig. 2. The g^2-order Feynman diagrams for πN scattering: *a)* the s-pole graph; *b)* the u-pole graph

where the symbol : : denotes the normal ordering*.

This result has been obtained in [18] (cf. Eq. (A.7) therein). The corresponding coefficients $V_{\pi N}(\mathbf{k}', \mathbf{p}', r'; \mathbf{k}, \mathbf{p}, r)$ that determine the pion-nucleon quasi-potential (see Eq. (4.5)) are equal to

$$\langle a_c{}^\dagger(\mathbf{k}') b_c{}^\dagger(\mathbf{p}', r')\Omega \mid K(\pi N \to \pi N) \mid a_c(\mathbf{k}) b_c(\mathbf{p}, r)\Omega\rangle \ . \tag{4.11}$$

They can be represented in the following covariant (Feynman-like) form

$$V_{\pi N}(\mathbf{k}_2, \mathbf{p}_2, r_2; \mathbf{k}_1, \mathbf{p}_1, r_1) = \frac{g^2}{(2\pi)^3}\delta(\mathbf{k}_2 + \mathbf{p}_2 - \mathbf{k}_1 - \mathbf{p}_1)\frac{1}{2\sqrt{\omega_{\mathbf{k}_2}\omega_{\mathbf{k}_1}}}\frac{m}{\sqrt{E_{\mathbf{p}_2}E_{\mathbf{p}_1}}}\times$$

$$\times\bar{u}(\mathbf{p}_2, r_2)\left\{\frac{1}{2}\left[\frac{1}{\not{k}_2 + \not{p}_2 + m} + \frac{1}{\not{k}_1 + \not{p}_1 + m}\right] + \right.$$

$$\left. + \frac{1}{2}\left[\frac{1}{\not{p}_2 - \not{k}_1 + m} + \frac{1}{\not{p}_1 - \not{k}_2 + m}\right]\right\}u(\mathbf{p}_1, r_1) \tag{4.12}$$

(see also Appendix in [32]).

In order to comment this expression, let us consider the Feynman graphs in Fig. 2 for the S-matrix elements of πN scattering. According to Feynman rules the four-momentum of the internal nucleon line in graph 2,*a* equals either the sum $k_1 + p_1$ of the incoming four-momenta or the sum $k_2 + p_2$ of the outgoing four-momenta. These sums are equal due to the energy and momentum conservation, viz., the S-matrix contains the multiplier

$$\delta(k_2 + p_2 - k_1 - p_1) = \delta(\omega_{\mathbf{k}_2} + E_{\mathbf{p}_2} - \omega_{\mathbf{k}_1} - E_{\mathbf{p}_1})\delta(\mathbf{k}_2 + \mathbf{p}_2 - \mathbf{k}_1 - \mathbf{p}_1). \tag{4.13}$$

Therefore, the Feynman propagator corresponding to the internal line can be written either as $(\not{k}_1 + \not{p}_1 + m)^{-1}$ or as $(\not{k}_2 + \not{p}_2 + m)^{-1}$. In the case of quasi-potential the energy conservation is not assumed (only the total three-momentun is conserved) and hence $k_1 + p_1$ is not necessarily equal to $k_2 + p_2$. The representation (4.12) shows that $V_{\pi N}$ includes the contribution associated with graph 2,*a* and

*Henceforth summation over the dummy spin indices is implied.

it can be obtained if we juxtapose to the internal nucleon line the half-sum $\frac{1}{2}\left[\frac{1}{\not k_2+\not p_2+m}+\frac{1}{\not k_1+\not p_1+m}\right]$.

In the case of S-matrix we juxtapose to the internal line in graph 2,*b* either the propagator $(\not p_2-\not k_1+m)^{-1}$ or $(\not p_1-\not k_2+m)^{-1}$. In the case of quasipotential $p_1-k_2\neq p_2-k_1$, in general, and the half-sum of these propagators does correspond to the internal line (it is the second half-sum in the curly brackets in the r.h.s. of Eq. (4.12)).

It follows from these observations that multiplying $V_{\pi N}$ by the factor $-2\pi\imath\delta(\omega_{\mathbf{k}_2}+E_{\mathbf{p}_2}-\omega_{\mathbf{k}_1}-E_{\mathbf{p}_1})$ we shall obtain the S-matrix elements for πN scattering in the g^2-order.

So, we have seen that the r.h.s. of Eq. (4.12) resembles the Feynman amplitudes being different from them in the two respects: i) the multiplier $\delta(\mathbf{k}_2+\mathbf{p}_2-\mathbf{k}_1-\mathbf{p}_1)$ is substituted instead of (4.13); ii) the above Feynman propagators are replaced by the corresponding half-sums.

4.3. Clothed Nucleon–Nucleon Eigenstates and Nucleon-Nucleon Quasipotential. Now, we consider the K_{ZA} eigenstates which belong to the NN sector, being superpositions of the kind,

$$\Phi_{NN}=\sum_r\int d\mathbf{p}_1d\mathbf{p}_2\Phi_{NN}(\mathbf{p}_1,r_1;\mathbf{p}_2,r_2)b_c{}^\dagger(\mathbf{p}_1,r_1)b_c{}^\dagger(\mathbf{p}_2,r_2)\Omega\ . \qquad (4.14)$$

A subset of such states with a definite momentum $\boldsymbol{P}$ is determined by Eq. (4.14) with the coefficients $\Phi_{NN}(1;2)\sim\delta(\mathbf{p}_1+\mathbf{p}_2-\boldsymbol{P})^*$. Obviously, $K_{ZA}\Phi_{NN}$ is the state vector of the same sector. In fact, the operators K_π, $K(\pi N\to\pi N)$, $K(\pi\bar N\to\pi\bar N)$, $K(N\bar N\to N\bar N)$, $K(\pi\pi\to N\bar N)$, and $K(N\bar N\to\pi\pi)$ involved in K_{ZA} do not contribute to $K_{ZA}\Phi_{NN}$ (see Eqs. (4.1) and (4.3)), and we find that K_{ZA} is reduced to the operator $K_N+K(NN\to NN)$. So, the eigenvalue equation $K_{ZA}\Phi^E_{NN}=E\Phi^E_{NN}$ yields the equation

$$[K_N+K(NN\to NN)]\Phi^E_{NN}=E\Phi^E_{NN} \qquad (4.15)$$

in the sector.

The corresponding equation for $\Phi^E_{NN}(1;2)$ (see Eq. (4.14)) can be derived from Eq. (4.15) by taking scalar products of both the parts of the latter with $\langle b_c{}^\dagger(\mathbf{p}_1,r_1)b_c{}^\dagger(\mathbf{p}_2,r_2)\Omega\,|$. Doing so, we get

$$(E-E_{\mathbf{p}_1}-E_{\mathbf{p}_2})\Phi^E_{NN}(\mathbf{p}_1,r_1;\mathbf{p}_2,r_2)=$$

$$=\int d\mathbf{p}_1'd\mathbf{p}_2'\tilde V_{NN}(\mathbf{p}_1,r_1,\mathbf{p}_2,r_2;\mathbf{p}_1',r_1',\mathbf{p}_2',r_2')\Phi^E_{NN}(\mathbf{p}_1',r_1';\mathbf{p}_2',r_2') \qquad (4.16)$$

*Here and sometimes below we use the evident abbreviations, viz., $1=(\mathbf{p}_1,r_1)$, etc.

with the properly symmetrized interaction (the quasipotential)

$$\tilde{V}_{NN}(1,2;1',2') = -\frac{1}{2}\left[V_{NN}(1,2;1',2') - V_{NN}(1,2;2',1') - \right.$$

$$\left. -V_{NN}(2,1;1',2') + V_{NN}(2,1;2',1')\right] \tag{4.17}$$

for the two clothed nucleons.

The two-body operator $K(NN \to NN)$ (see Eq. (4.4)) is generated by the second term in the curly brackets of Eq. (A.14) and its H.c.:

$$K(NN \to NN) = \frac{1}{2} : \left\{ [\mathcal{R}, \mathcal{V}^\dagger]_{NN} + [\mathcal{R}, \mathcal{V}^\dagger]^\dagger_{NN} \right\} : \tag{4.18}$$

One should note that the coefficients $V_{NN}(1,2;1',2')$ in Eq. (4.4) are not in the one-to-one correspondence with $K(NN \to NN)$, viz., they can be changed without altering the latter. For instance, the property

$$b(1)b(2) = -b(2)b(1) \tag{4.19}$$

enables one to replace $V_{NN}(1,2;1',2')$ in Eq. (4.4) by $-V_{NN}(1,2;2',1')$, and so on.

Moreover, the operator $K(NN \to NN)$ remains unaltered when adding to V_{NN} arbitrary functions $S_L(1,2;1',2')$ or $S_R(1,2;1',2')$ which are symmetrical under the transpositions $1 \leftrightarrow 2$ or $1' \leftrightarrow 2'$, respectively. The above-mentioned replacement $V_{NN}(1,2;1',2') \to -V_{NN}(1,2;2',1')$ is equivalent to the replacement

$$V_{NN}(1,2;1',2') \to V_{NN}(1,2;1',2') + S_R(1,2;1',2') \tag{4.20}$$

with $S_R(1,2;1',2') = -V_{NN}(1,2;1',2') - V_{NN}(1,2;2',1')$.

A distinctive feature of the coefficient (4.17) is its invariance with respect to the transformation (4.20) with arbitrary S_R.

After these notations, we write down one of the possible expressions for V_{NN}, that can be obtained using Eq. (4.18),

$$V_{NN}(1,2;1',2') = -\frac{1}{2}\int d\mathbf{k}\left\{ \frac{1}{E_{\mathbf{p}_1} - E_{\mathbf{p}'_1} - \omega_{\mathbf{k}}} + \frac{1}{E_{\mathbf{p}'_2} - E_{\mathbf{p}_2} - \omega_{\mathbf{k}}} \right\} \times$$

$$\times V_{11}^{-\mathbf{k}}(\mathbf{p}_1 r_1; \mathbf{p}'_1 r'_1) V_{11}^{\mathbf{k}}(\mathbf{p}_2 r_2; \mathbf{p}'_2 r'_2) \ . \tag{4.21}$$

The respective quasipotential is

$$\tilde{V}_{NN}(1,2;1',2') = -\frac{g^2}{(2\pi)^3}\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2)\frac{m^2}{2\sqrt{E_{\mathbf{p}_1} E_{\mathbf{p}_2} E_{\mathbf{p}_1\prime} E_{\mathbf{p}_2\prime}}} \times$$

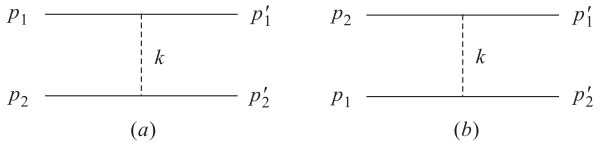

Fig. 3. The one-pion-exchange Feynman diagrams for NN scattering

$$\times\bar{u}(1)\gamma_5 u(1')\frac{1}{2}\left\{\frac{1}{(p_1-p_1')^2-\mu^2}+\frac{1}{(p_2-p_2')^2-\mu^2}\right\}\bar{u}(2)\gamma_5 u(2') \\ -(1\leftrightarrow 2). \tag{4.22}$$

Expression (4.22) is the NN part of an one-boson-exchange interaction derived via the Okubo transformation method in [25] (cf. [43]). The potential $\tilde{V}_{NN}$ consists of the direct term written explicitly and the exchange term $(1\leftrightarrow 2)$. In order to obtain the latter one needs to replace $\mathbf{p}_1, r_1$ by $\mathbf{p}_2, r_2$ and $\mathbf{p}_2, r_2$ by $\mathbf{p}_1, r_1$ in the former.

As has been pointed out in [25]*, a distinctive feature of the potential is the appearance of a covariant (Feynman-like) «propagator»

$$\frac{1}{2}\left\{\frac{1}{(p_1'-p_1)^2-\mu^2}+\frac{1}{(p_2'-p_2)^2-\mu^2}\right\}, \tag{4.23}$$

where $p=(E_{\mathbf{p}},\mathbf{p})$ is the nucleon four-momentum. On the energy shell, that is, when

$$E_i\equiv E_{\mathbf{p}_1}+E_{\mathbf{p}_2}=E_{\mathbf{p}_1'}+E_{\mathbf{p}_2'}\equiv E_f, \tag{4.24}$$

the r.h.s. of Eq. (4.23) becomes the genuine Feynman propagator which appears when evaluating the S-matrix for NN scattering in the g^2-order. The respective graphs are displayed in Fig. 3. Like $V_{\pi N}$ the quasipotential $\tilde{V}_{NN}$ can be associated with these Feynman graphs being different from the corresponding Feynman amplitude in the two respects, viz., $\tilde{V}_{NN}$ does not contain $\delta(E_{\mathbf{p}_1}+E_{\mathbf{p}_2}-E_{\mathbf{p}_1'}-E_{\mathbf{p}_2'})$, and «propagator» (4.23) now corresponds to the internal meson line in graph 3,*a*. A more extended analysis of this observation has been given in [25].

In conclusion, one should note that all the quasipotentials are nonlocal since the vertices and propagators in Eqs. (4.12) and (4.22) are dependent not only on the relative three-momenta involved but also on their total three-momentum. They include the nonstatic (recoil) effects in all orders of the so-called $\frac{1}{c^2}$ expansion [44].

*There one can find another representation of the nucleon–nucleon quasipotential, which resembles the expressions of old-fashioned perturbation theory (see, e.g., Ch. 13 in [15]).

4.4. Other Clothed Eigenstates of Meson-Fermion System. Up to now we have focused upon the clothed πN and NN states. Let us discuss other clothed states.

If we start with the same «zeroth» approximation to the total Hamiltonian, our description of clothed $\pi\bar{N}$ and $\bar{N}\bar{N}$ states will be very similar to that given for πN and NN states. Actually, it is the case where one has to deal with the charge-conjugated states. Here, we mean the nucleon–antinucleon conjugation.

A different situation holds in the case of clothed fermion–antifermion and two-meson states. In fact, the operator $K_4^{(2)}$ contains the interactions $K(\pi\pi \leftrightarrow N\bar{N})$. Therefore, superpositions of the $\pi\pi$ states ${a_c}^\dagger {a_c}^\dagger\Omega$ and the $N\bar{N}$ states ${b_c}^\dagger {d_c}^\dagger\Omega$ taken separately cannot be K_{ZA} eigenvectors. So, one has to consider the eigenstates of a mixed kind,

$$\Phi = \int d\mathbf{k}_1 d\mathbf{k}_2 \Phi_{\pi\pi}(\mathbf{k}_1;\mathbf{k}_2) {a_c}^\dagger(\mathbf{k}_1) {a_c}^\dagger(\mathbf{k}_2)\Omega +$$

$$+ \int d\mathbf{p}_1 d\mathbf{p}_2 \Phi_{N\bar{N}}(\mathbf{p}_1, r_1; \mathbf{p}_2, r_2) {b_c}^\dagger(\mathbf{p}_1, r_1) {d_c}^\dagger(\mathbf{p}_2, r_2)\Omega \ . \qquad (4.25)$$

Calculation of the scalar products of $K_{ZA}\Phi = E\Phi$ with $\langle {a_c}^\dagger {a_c}^\dagger\Omega \mid$ and $\langle {b_c}^\dagger {d_c}^\dagger\Omega \mid$ leads to a set of coupled equations for the coefficients $\Phi_{\pi\pi}$ and $\Phi_{N\bar{N}}$. Of course, one may obtain separate equations for each of them. Thereat, the eigenvalue equation for $\Phi_{N\bar{N}}$ will involve some terms of the g^4-order. Obviously, to be consistent they should be disregarded within the ZA considered.

In the analogous manner one can study the eigenvalue problem for clothed three-nucleon and more complicated states. However, handling with K_{ZA}, we enter into the $3N$-problem only with the two-body interaction $K(NN \to NN)$. It would be interesting to take into account the three-body (six-operator) interactions (irreducible to two-body ones) that are present in the total Hamiltonian K starting from the term $[R,[R,[R,V]]] \sim g^4$ not explicitly written in Eq. (2.23).

5. POSSIBLE MODIFICATIONS OF THE CLOTHING APPROACH

5.1. Heitler's Unitary Transformation. The «clothing» in Sec. 2 has been realized in the framework of the Schrödinger picture, the «bare» and «clothed» operators being the Schrödinger ones. Heitler in his book [5] discussed the corresponding UT's in the framework of the interaction picture. To establish the relation with Heitler's approach we shall derive Heitler's equation (which determines his UT) starting from our equation (2.19) of Sec. 2:

$$W(\alpha)[H_F(\alpha) + H_I(\alpha)]W^\dagger(\alpha) = K(\alpha) = K_0(\alpha) + K_I(\alpha). \qquad (5.1)$$

Remind that the free part $K_0(\alpha)$ of the total Hamiltonian $K(\alpha)$ (expressed in terms of the clothed operators α) is equal to $H_F(\alpha)$, $H_F(\alpha)$ being given by

Eq. (2.6) in which bare operators a, b, and d are replaced by the clothed ones a_c, b_c, and d_c. Now, let us write Eq. (5.1) in terms of interaction picture operators defined, e.g., as

$$\alpha_p(t) = \mathrm{e}^{iK_0(\alpha)t}\alpha_p\mathrm{e}^{-iK_0(\alpha)t}. \tag{5.2}$$

For this purpose multiply both parts of Eq. (5.1) by $\exp(iK_0t)$ from the left and by $\exp(-iK_0t)$ from the right and use the notation

$$A(t) = \mathrm{e}^{iK_0t}A(\alpha)\mathrm{e}^{-iK_0t} = A(\mathrm{e}^{iK_0t}\alpha\mathrm{e}^{-iK_0t}) = A(\alpha(t)).$$

Then we obtain,

$$W(t)[H_F(t) + H_I(t)]W^\dagger(t) = K_0(t) + K_I(t). \tag{5.3}$$

Multiply both parts of this equation by $W^\dagger(t)$ from the left and use $W^\dagger W = 1$ and $K_0(t) = H_F(t)$ (see above). We get

$$[H_F(t), W^\dagger(t)] = -H_I(t)W^\dagger(t) + W^\dagger(t)K_I(t). \tag{5.4}$$

The l.h.s. of this equation is equal to $-i\frac{\partial}{\partial t}W^\dagger(t)$ (see, e.g., Eq. (11.52) in [15]). So, we find,

$$H_I(t)W^\dagger(t) - i\frac{\partial}{\partial t}W^\dagger(t) = W^\dagger(t)K_I(t). \tag{5.5}$$

This equation coincides with Heitler's equation written in [5] (Ch. 4, between Eqs. (15.6) and (15.7_1)). The relation of Heitler's notations and ours is

$$S = W^\dagger, \quad K = K_I, \quad H = H_I.$$

So, Heitler's basic equation is equivalent to our Eq. (2.19) or (5.1). But his goal differs from the goal of «clothing». He requires that K_I must not contain interaction terms which give rise to virtual processes. By definition, the latter can proceed in spite of the inequality of energies of the initial and final states (here «energy» means an eigenvalue of the free part of the total Hamiltonian). Our bad terms also lead to virtual processes (e.g., $\Omega \to \pi\bar{N}N$ or $N \to N\pi$) but there are many other virtual processes not generated by bad terms (e.g., $\pi\pi \to \bar{N}N$ or $\pi N \to \pi\pi N$ at low initial energies).

Of course, Heitler's requirement can be imposed also in the Schroedinger picture and this has been done by Sato et al. [32, 35].

5.2. Heitler–Sato Approach Versus the Clothing One. We shall show here that under a condition all bad terms produce virtual processes. As there are many virtual processes which are not induced by bad terms, one may state that the Heitler–Sato condition is stronger than the bad terms elimination requirement.

Let us remind the exact definition of bad terms. They are either two-, three-, ... operator terms which contain only creation operators (and do not

contain destruction operators) or three-, four-, ... operators containing only one destruction operator. We call bad also the terms which are Hermitian conjugated to the above-mentioned. Using the notion of the class defined in Sec. 2 one may define the bad terms as terms of the class $[n, 0]$ and $[n, 1]$, $n \geq 2$ and their H.c. Note that two-operator terms of the kind $a^\dagger(k)a(k)$ are not attached to the bad ones.

Bad interaction terms are responsible for the processes

$$\begin{array}{lcll} \text{no particles} & \leftrightarrow & 2, 3, \ldots & \text{particles} \\ \text{one-particle} & \leftrightarrow & 2, 3, \ldots & \text{particles.} \end{array} \tag{5.6}$$

This property also may be considered as the bad terms definition. The processes (5.6) will be called bad below.

Let us prove the Statement: All bad processes are virtual ones under a condition on the masses of interacting particles. Indeed, energy is evidently not conserved in the bad process «vacuum $\rightarrow$ several particles»: the initial energy is zero while the final state energy cannot be less than the sum of the final masses. Further, consider the bad process $a_j \rightarrow a_1 + a_2 + \ldots$, in which the particle a_j with the mass m_j converts into particles with masses $m_1, m_2, \ldots$. In the particle a_j rest frame the initial energy is m_j. The energy is not conserved trivially if sum of the masses $\Sigma_i m_i$ of the final particles exceeds m_j. The energy is conserved if $m_j > \Sigma_i m_i$ and final particles possess nonzero momenta. The set of inequalities $m_j < \Sigma_i m_i$, $\forall_j$ is the very Statement condition on the particle masses.

In the Yukawa model the bad processes $N \rightarrow N\pi$, $N \rightarrow N\bar{N}N$, $N \rightarrow N\pi\pi$, etc., are certainly virtual, the process $\pi \rightarrow \bar{N}N$ being virtual if $\mu < 2m$. Under this condition all the bad processes generated with the Yukawa model are virtual.

Let us consider the simplest virtual processes of the Yukawa model which are of the order g^1, namely $\Omega \rightarrow N\bar{N}\pi$, $N \rightarrow N\pi$, $\pi \rightarrow \bar{N}N$ (if $\mu < 2m$). They coincide with the (simplest) bad processes of the order g^1. The corresponding three-operator interaction terms are of the order g^1 and can be removed by the unitary operator $W = \exp R$, where R is a three-operator expression of the order g^1 (see Sec. 2).

6. THE OKUBO BLOCK-DIAGONALIZATION METHOD

Here, following [7] we regard the UT $H \rightarrow H_U \equiv U^\dagger H U$ that makes the Hamiltonian H (generally speaking, an Hermitian operator) block-diagonal (cf. our brief discussion in Introduction). After the transformation the primary H eigenvalue problem is reduced to the diagonalization of separate neardiagonal blocks of H_U.

It is well known that the transformation $H \rightarrow H_U$ can be interpreted either as the connection between the two matrices of one and the same operator H with

respect to the different orthonormal bases, or as the relationship between the two operators H and H_U unitarily connected via the operator U. We shall start from the first point of view.

6.1. UT as Change of Basis. Let $H_{n'n} \equiv \langle n'|H|n\rangle\,(\forall n', n)$ be the H matrix with respect to a complete set of orthonormal vectors $|n\rangle$ and $H_{\nu'\nu} \equiv (\nu'|H|\nu)\,(\forall\nu', \nu)$ represents H in another orthonormal basis $|\nu)$. The indices $n(\nu)$ can take on discrete or/and continuous values. One has (see, e.g., [45], Ch.I)

$$H_{\nu'\nu} \equiv (\nu'|H|\nu) = \mathbf{S}_{n'}\mathbf{S}_n\,(\nu'|n'\rangle\langle n'|H|n\rangle\langle n|\nu)\,, \tag{6.1}$$

where $\mathbf{S}_n$ denotes a sum or/and an integral over n.

The r.h.s. of Eq. (6.1) can be written as the matrix product

$$\mathbf{S}_{n'}\mathbf{S}_n\,(U^\dagger)_{\nu'n'}H_{n'n}U_{n\nu} \equiv (U^\dagger HU)_{\nu'\nu}\,, \tag{6.2}$$

where we have introduced the notation $U_{n\nu} \equiv \langle n|\nu)$. Then

$$(U^\dagger)_{\nu'n'} \equiv U^*_{n'\nu'} = \langle n'|\nu')^* = (\nu'|n'\rangle.$$

The transition matrix U is unitary in the following sense:

$$(U^\dagger U)_{\nu'\nu} \equiv \mathbf{S}_n\,(\nu'|n\rangle\langle n|\nu) = \delta_{\nu'\nu} \tag{6.3a}$$

and

$$(UU^\dagger)_{n'n} \equiv \mathbf{S}_\nu\,\langle n'|\nu)(\nu|n\rangle = \langle n'|n\rangle = \delta_{n'n}\ . \tag{6.3b}$$

Recall that all these basis vectors are orthonormal.

Whenever U is given one can express $|\nu)$ in terms of $|n\rangle$,

$$|\nu) = \mathbf{S}_n\,|n\rangle\,U_{n\nu}\,. \tag{6.4}$$

One should point out that the spectrum of indices n enumerating the vectors $|n\rangle$ needs not to be identical to the spectrum of indices ν enumerating $|\nu)$. For example, let us consider the Hamiltonian H_{os} of one-dimensional quantum oscillator, whose matrix in the coordinate representation can be made diagonal by using the Hermite functions $\psi_\nu(x)\,(\nu = 0, 1, 2, \dots)$:

$$(\nu'|H_{os}|\nu) = \int\limits_{-\infty}^{\infty} dx' \int\limits_{-\infty}^{\infty} dx\,\psi^*_{\nu'}(x')\,\langle x'|H_{os}|x\rangle\,\psi_\nu(x) \sim \delta_{\nu'\nu} \tag{6.5}$$

(cf. Eqs. (6.1)–(6.2)). In this case the transition matrix $\langle x|\nu) = \psi_\nu(x)$ is not «square» because its columns (rows) are enumerated by the integer (continuous) numbers. Such a matrix can be called rectangular. Note that in a finite-dimensional space the transition matrix always is square. This exemplifies that one cannot, in general, diagonalize H by means of a square matrix.

In this connection, one should note that the diagonalization scheme considered by Okubo and exposed below is not aimed at a perfect solution of H-eigenvalue problem. Rather, it sets a more humble task, viz., to find (approximately) some of its eigenvalues and eigenvectors. Okubo [7] suggested a realization of the scheme via a square matrix U.

6.2. Block-Diagonalization in Matrix Form. In accordance with [7] we require that the H matrix in the new basis

$$H_{\nu'\nu} = (U^\dagger\, H\, U)_{\nu'\nu} \tag{6.6}$$

should have the block-diagonal form,

$$U^\dagger H U \equiv K = \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix} \tag{6.7}$$

with the two blocks K_{11} and K_{22} the meaning of which will be clarified a little later. This requirement must determine the matrix U that in its turn enables one to construct the new basis vectors $|\nu)$ (see Eq. (6.4)).

Now, following Okubo, we shall confine ourselves to finding such a square matrix U that $|\nu) = U\,|n\rangle$. In other words, it is assumed that one can set an one-to-one correspondence between the indices n and ν.

First of all, let us turn to the definition of K_{11}. It is a block of the matrix K with elements that are enumerated by indices ν_1 (or n_1) which belong to a subset of all ν (or n) values. The K_{22} elements are enumerated by the remaining ν values denoted through ν_2. Let us give examples of the ν_1 choice.

One may take one value of ν as ν_1, viz., the index of the vacuum state. Hence, K_{11} has one element. If we are able to find U which leads to Eq. (6.7) then we can construct a normalized H eigenstate, namely the physical vacuum $|0) = \mathbf{S}_n\,|n\rangle\, U_{n0}$. After this step we consider the block K_{22} as a starting matrix for the subsequent block-diagonalization, viz., to introduce a new set of ν_1' (e.g., let ν_1' be indices of the one-particle H_F eigenstates), to find a new U'. At the next stage one can enumerate elements of the recurrent block 11 by indices of two-particle states, e.g., the states «two nucleons, no mesons» (this is Okubo's example, see Sec. 2 in [7]). One more choice of K_{11} will be discussed in Subsec. 6.5.

The option of ν_1 or n_1 allows one to divide H into the four blocks: H_{11} with elements $\langle n_1'|H|n_1\rangle$, H_{22} (elements $\langle n_2'|H|n_2\rangle$), H_{12} (elements $\langle n_1|H|n_2\rangle$), and $H_{21} = {H_{12}}^\dagger$. The matrix U can be represented analogously. Keeping this in mind the l.h.s. of Eq. (6.7) can be rewritten as the product of matrices composed of the blocks described above (see, e.g., Ch. 0.7 in [47] and Ch. 1.6 in [48]),

$$\begin{pmatrix} {U_{11}}^\dagger & {U_{21}}^\dagger \\ {U_{12}}^\dagger & {U_{22}}^\dagger \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} K_{11} & 0 \\ 0 & K_{22} \end{pmatrix} . \tag{6.7$'$}$$

One can consider Eq. (6.7′) as the equation for U. Okubo suggested to seek its solution in the class of unitary matrices of the kind (see Note underneath)

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} (I_1 + A^\dagger A)^{-\frac{1}{2}} & -A^\dagger (I_2 + AA^\dagger)^{-\frac{1}{2}} \\ A(I_1 + A^\dagger A)^{-\frac{1}{2}} & (I_2 + AA^\dagger)^{-\frac{1}{2}} \end{pmatrix} , \quad (6.8)$$

where $I_1(I_2)$ is the unit matrix for the subset of indices $n_1(n_2)$, and A is an rectangular matrix with elements $A_{n_2 n_1}$ of the U_{21} kind, which should be determined. Under this convention the unit matrix for the full set of indices has the block structure

$$1 = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} , \quad (6.9)$$

and we have the properties

$$AI_1 = I_2 A = A , \quad (6.10a)$$

$$[A^\dagger A, I_1] = 0 . \quad (6.10b)$$

The matrices $A^\dagger A$ (of the U_{11} kind) and $AA^\dagger$ (of the U_{22} kind) are square, hermitian and positively definite, so that the square roots $\sqrt{I_1 + A^\dagger A}$ and $\sqrt{I_2 + AA^\dagger}$ can be defined (see, e.g., Ch. 5.8 in [45]). It is readily verified that the r.h.s. of (6.8) meets left unitarity $U^\dagger U = 1$. Other unitarity condition $UU^\dagger = 1$ can be proved by means of equality $Af(A^\dagger A) = f(AA^\dagger)A$ that is valid if $f(x)$ is a polynomial or a series of x.

The requirement for block-diagonalization, $K_{21} = 0$ (or $K_{12} = 0$), gives the equation

$$H_{21} + H_{22}A - AH_{11} - AH_{12}A = 0 \quad (6.11)$$

for A determination. This equation is equivalent to the condition (13) from [7].

Equation (6.11) is nonlinear and it can be solved exactly only in a few simple cases (see Sec. 6 in [7] and Appendix C). For realistic field models one has to develop a perturbative method in order to find A (see Subsec. 6.5).

Note. Generally speaking, the left unitarity for a matrix U with the non-singular block U_{11} enables to express blocks $U_{ij}(i, j = 1, 2)$ through the matrix A such that $U_{21} = AU_{11}$ and blocks $S_{11} = S_1$ and $S_{22} = S_2$ of an arbitrary matrix S,

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$$

with ${S_1}^\dagger S_1 = I_1$ and ${S_2}^\dagger S_2 = I_2$ (cf. Eqs. (10) in [7]). The corresponding matrix U_{Okubo} (i.e., its representation after Okubo) can be written as the product $US = U_{\mathrm{Okubo}}$ where U is given by (6.8). However, since the UT via S conserves the block-diagonal structure of K it is sufficient to consider only the form (6.8). Of

course, Okubo's representation is not the most general form for unitary matrices. The simplest exception is the 2×2 matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

6.3. Block-Diagonalization with Projection Operators. Here the UT $H' = U^\dagger H U$ is interpreted as a relation between the different operators H and H'*, where U is a unitary operator. By definition, the operator is a linear one-to-one mapping of a Hilbert space $\mathcal{H}$ onto itself, i.e., an isomorphism of $\mathcal{H}$. In particular, it transforms the orthonormal basis vectors $|n\rangle \in \mathcal{H}$ into the vectors $|n) = U|n\rangle$ of other orthonormal basis $\in \mathcal{H}$. Therefore, we can write

$$\langle n'|H'|n\rangle = \langle n'|U^\dagger H U|n\rangle = (n'|H|n) \ , \tag{6.12}$$

Eq. (6.12) sets up close links of $H' = U^\dagger H U$ with the UT considered in Subsecs. 6.1 and 6.2.

One can deal with $H' = U^\dagger H U$ in the same manner as in Subsec. 6.2 introducing matrices of H, U, $H^\dagger$ with respect to a basis $|n\rangle$ and representing them in a block form, Eq. (6.7′). But one can realize the Hamiltonian block-diagonalization without reference to any basis on $\mathcal{H}$. We shall introduce (cf. [7]) the projection operators η_1 and $\eta_2 = 1 - \eta_1$ onto a subspace $\mathcal{H}_1 \subset \mathcal{H}$ and its complement $\mathcal{H}_2$ such that $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$*. Mathematically strict definitions of projection operators and their properties in a Hilbert space can be found in [46] (Ch.2). Of great importance for us is the property

$$\eta_i \eta_k = \eta_k \eta_i = \delta_{ik} \eta_i. \tag{6.13}$$

$$(i, k = 1, 2).$$

Then, for any operator O in $\mathcal{H}$ one can write

$$O = (\eta_1 + \eta_2)\, O\, (\eta_1 + \eta_2) = \sum_{i,j=1,2} O^{ij} \ , \tag{6.14}$$

where

$$O^{ij} = \eta_i\, O\, \eta_j \tag{6.15}$$

with

$$\eta_i\, O^{kj} = \delta_{ik} O^{ij}, O^{kj}\, \eta_l = O^{kl} \delta_{jl} \ . \tag{6.16}$$

*Unlike the UT $K = W\, H\, W^\dagger$ in Subsec. 2.3 that represents the same operator H.

*Of course, the particular option of $\mathcal{H}_1$ depends on the nature of the problem (see Appendix C and Subsec. 6.5).

In terms of such decompositions the product of the two operators $A = \sum_{ij} A^{ij}$ and $B = \sum_{ij} B^{ij}$ can be written as

$$A\,B = \sum_{ijk} A^{ij} B^{jk} \ , \tag{6.17}$$

so that

$$(A\,B)^{ik} = \sum_{j} A^{ij} B^{jk} \ . \tag{6.18}$$

By using the rule we find

$$H' = U^{\dagger} H U = \sum_{imjk} (U^{\dagger})^{mi} H^{ij} U^{jk} \tag{6.19a}$$

or

$$H' = \sum_{imjk} (U^{im})^{\dagger} H^{ij} U^{jk} \ . \tag{6.19b}$$

Remind that the projectors η_i are hermitian , i.e., ${\eta_i}^{\dagger} = \eta_i$.

One can set an one-to-one correspondence between the operators H^{ij} and the matrix blocks H_{ij} introduced in Subsec. 6.2. Actually, each operator H^{ij} $(i, j = 1, 2)$ acts in the full Hilbert space $\mathcal{H}$ but its matrix with respect to the basis $\{|n\rangle\}$ has merely one nonzero block H_{ij} with the three remaining blocks being zero. Analogous relation takes place between U^{ij} and U_{ij}.

Further, the operators U^{ij} in Eqs. (6.19) can be expressed through an arbitrary operator $\mathcal{A}$ of the kind 21, i.e., $\mathcal{A} = \eta_2 \mathcal{A} \eta_1$. For instance, one can write (cf. Eq. (6.8))

$$U^{21} = \mathcal{A} U^{11} = \mathcal{A}(1 + \mathcal{A}^{\dagger}\mathcal{A})^{-\frac{1}{2}} \eta_1 = \mathcal{A}(1 + \mathcal{A}^{\dagger}\mathcal{A})^{-\frac{1}{2}} \ . \tag{6.20}$$

Okubo's requirement for decoupling the two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ means that it must be ${H'}^{21} = 0$ or according to Eq. (6.19a)

$$\sum_{ij} (U^{\dagger})^{2i} H^{ij} U^{j1} = 0. \tag{6.21}$$

It leads to the nonlinear equation for $\mathcal{A}$

$$\eta_2 \{H + [H, \mathcal{A}] - \mathcal{A} H \mathcal{A}\} \eta_1 = 0. \tag{6.22}$$

As a matter of fact, it is the same equation as Eq. (6.11), viz., one may consider the latter as the record of Eq. (6.22) for the only nonzero block A of the operator $\mathcal{A}$.

The solution of Eq. (6.22), if it exists (cf. the discussion in [21]), yields the Hermitian operator

$${H'}^{11} = \eta_1 (1 + \mathcal{A}^{\dagger}\mathcal{A})^{-\frac{1}{2}} (1 + \mathcal{A}^{\dagger}) H (1 + \mathcal{A}) (1 + \mathcal{A}^{\dagger}\mathcal{A})^{-\frac{1}{2}} \eta_1. \tag{6.23}$$

The resultant operator H' given by Eqs. (6.19) (see also Eq. (6.23)) has a block diagonal structure that simplifies determination of its eigenvalues and eigenvectors. However, $H' \neq H$ and therefore H' eigenvectors are not H eigenvectors. Recall that our UT is aimed at solving the H eigenvector problem (at least, its partial solution). But if we have succeeded in finding H'^{11} eigenvectors one can easily get the corresponding H eigenvectors. Actually, if we have $H'^{11}\Psi_1 = E\Psi_1$ ($\Psi_1 \in \mathcal{H}_1$), then Ψ_1 is also H' eigenvector,

$$H'\Psi_1 = (H'^{11} + H'^{22})\Psi_1 = E\Psi_1. \tag{6.24}$$

It follows from $U^\dagger H U \Psi_1 = E\Psi_1$ that $U\Psi_1$ is H eigenvector,

$$H\,U\Psi_1 = E\,U\Psi_1. \tag{6.25}$$

The obtained relation between H' and H eigenvectors can be represented in another form. Let us expand the H' eigenvector in the basis vectors $|n\rangle$,

$$|\Psi_1\rangle = \mathbf{S}_n |n\rangle c_n.$$

The corresponding H eigenvector $U\Psi_1$ has the same expansion coefficients c_n with respect to the other basis $|n) = U|n\rangle$,

$$U\Psi_1 = \mathbf{S}_n U|n\rangle c_n = \mathbf{S}_n |n) c_n. \tag{6.26}$$

Exact solutions of the decoupling equation (6.22) can be derived for a few simple cases (see, e.g., Sec. 6 in [7]). One of them is the so-called scalar field model (see Ch. 12 in [15]). The respective solution is very instructive having many attributes of more realistic field models. It is given in Appendix C.

6.4. New Creation–Destruction Operators within the UT Method. Comparison with the Clothing Procedure. As before (see Sec. 2) we consider that the original field Hamiltonian H is a function (functional) of the bare creation–destruction operators. We denote their set by the same symbol a, a_p being one of them. Moreover, one may assume that the unitary operator U also is a function of a. Then, $H' = U^\dagger H U$ may be written as

$$H'(a) = U^\dagger(a) H(a) U(a). \tag{6.27}$$

In order to compare the Okubo approach and the clothing procedure developed in Sec. 2 let us introduce the set $\tilde{a}$ of new creation–destruction operators (with the denotation $\tilde{a}_p$ for one of them) defined as

$$\tilde{a}_p = V(a) a_p V^\dagger(a) \quad \forall p\ , \tag{6.28}$$

or

$$a_p = V^\dagger(\tilde{a}) \tilde{a}_p V(\tilde{a})\ , \tag{6.29}$$

where $V(a) = V(\tilde{a}) \neq 1$ is an arbitrary unitary operator (not necessarily coincident with $U(a)$).

Applying the UT V to both the sides of Eq. (6.27), we have

$$V(a)\, H'(a)\, V^{\dagger}(a) = H'(\tilde{a}) = U^{\dagger}(\tilde{a})\, H(\tilde{a})\, U(\tilde{a}). \tag{6.30}$$

Further, by means of (6.29) the total Hamiltonian can be expressed (cf. the derivation of Eq. (2.17)) in terms of the new operators $\tilde{a}$,

$$H(a) = H(V^{\dagger}(\tilde{a})\, \tilde{a}\, V(\tilde{a})) = V^{\dagger}(\tilde{a})\, H(\tilde{a})\, V(\tilde{a}). \tag{6.31}$$

One should stress that the operators $H'(\tilde{a})$ and $H(\tilde{a})$ are different from $H'(a)$ and $H(a)$, respectively, if $V \neq 1$. However, if $V = U$, then $H'(\tilde{a})$ turns out to be equal to the starting Hamiltonian $H(a)$. Actually, in this case the r.h.s of Eq. (6.31) gets to be equal to $H'(\tilde{a})$ in accordance with Eq. (6.30), so that

$$H(a) = H'(\tilde{a}) = U^{\dagger}(\tilde{a}) H(\tilde{a}) U(\tilde{a}). \tag{6.31$'$}$$

The option $V = U = W^{\dagger}$ gives rise to the clothed operators $\tilde{a} = \alpha = W^{\dagger} a W$ (see Subsec. 2.3), and then $H'(\alpha) = K(\alpha) = W H(\alpha) W^{\dagger}$.

Thus, our consideration shows how the UT $H \to H' = U^{\dagger} H U$ can be reduced to a transformation of clothing type. Of course, there are distinctions between the Okubo approach and the procedure shown in Sec. 2. They are due to those purposes which are inherent to each of them. We shall return to this point in Subsec. 6.6. However, let us note here that although the operator $U_{\mathrm{Okubo}}(a)$ determined by solving the decoupling equation (6.22) and the operator $U_{\mathrm{clothing}}(a) = W^{\dagger}(a)$ are different functions of a one cannot *á priori* exclude a resemblance or even the perfect coincidence of some approximations to them.

Note one more definition of $\tilde{a}$ by the relations

$$(n'|\tilde{a}_p|n) = \langle n'|\, a_p|n\rangle \quad \forall n', n, p\ , \tag{6.32}$$

where $|n) = U\,|n\rangle$ (cf. the beginning of Subsec. 6.3). In particular, Eq.(6.32) means that the matrix elements of the new meson destruction operator $\tilde{a}(\mathbf{k})$ with respect to the new vectors $|n)$ are equal to the corresponding matrix elements of the bare meson destruction operator $a(\mathbf{k})$ with respect to the old vectors $|n\rangle$. It is clear that this definition with $U = U\,(a)$ is equivalent to (6.28).

6.5. Perturbative Construction of Okubo's Unitary Transformation. Elimination of Mesonic Degrees of Freedom. Here we present the approximate solution of Okubo's equation and determination of the block H'^{11} that have been given in [25]. Just as in that paper let us consider the system of fermions (nucleons) and mesons (pions) with Yukawa-type interaction linear in the meson field (see Sec. 2). In this model the basis states $|n\rangle$ are enumerated by the two indices

$|n\rangle = |mf\rangle$, viz., $m(f)$ enumerates the meson (fermion) states, $|0_m f\rangle$ being the state without mesons (the $N\bar{N}$ contents may be arbitrary). In accordance with the Okubo idea we decompose the full space $\mathcal{H}$ of meson-nucleon states into two subspaces (sectors), namely, the fermion (nucleon) sector $\mathcal{H}_0$ which is composed of no-meson states, being spanned onto the subset $\mid 0_m f\rangle$, and its orthogonal complement $\mathcal{H}_{\rm comp}$ that consists of the states with nonzero meson number. The projection operator η_1 into $\mathcal{H}_0$ * can be constructed as

$$\eta_1 = \mathbf{S}_f \mid 0_m f\rangle\langle 0_m f \mid \ . \tag{6.33}$$

By definition,

$$a(\mathbf{k}) \mid 0_m f\rangle = 0 \quad \forall f, \mathbf{k}\ , \tag{6.34}$$

and therefore

$$a(\mathbf{k})\eta_1 = \eta_1 a^\dagger(\mathbf{k}) = 0. \tag{6.35}$$

The operator of interest H'^{11} has the following matrix structure (cf. the note after Eqs. (6.19)):

$$H'^{11} = \begin{pmatrix} H'_{11} & 0 \\ 0 & 0 \end{pmatrix}\ , \tag{6.36}$$

where the block H'_{11} consists of the elements $\langle 0_m f' \mid H' \mid 0_m f\rangle$, $\forall f', f$. At the same time the matrix

$$H'^{22} = \begin{pmatrix} 0 & 0 \\ 0 & H'_{22} \end{pmatrix} \tag{6.37}$$

contains the elements $\langle m'f' \mid H' \mid mf\rangle$, $\forall m' \neq 0_m \neq m$, $\forall f', f$. For brevity, henceforth the index 0_m of the no-meson state will be replaced by 0.

In order to simplify the subsequent equations let us separate the matrix of any operator O that acts in $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_{\rm comp}$ into the subblocks $[O_{m'm}]$ such that

$$[O_{m'm}]_{f'f} = \langle m'f' \mid O \mid mf\rangle. \tag{6.38}$$

So, the elements of these subblocks are marked by the fermionic indices f. The subblocks can be called fermionic. The block H_{11} coincides with $[H_{00}]$, etc. The rectangular matrix A, which determines U in Eq. (6.8), consists of the subblocks $[A_{m'0}]$ with $m' \neq 0$, i.e., the submatrices $[A_{\mathbf{k}0}]$ with $m' = \mathbf{k}$ for one-meson states marked by the momentum $\mathbf{k}$, the submatrices $[A_{\mathbf{k}_1\mathbf{k}_2;0}]$ with $m' = (\mathbf{k}_1\mathbf{k}_2)$ for two-meson states, and so on.

Now, the key equation $H' = U^\dagger H U$ can be rewritten as the set of equations for the subblocks $[U_{m'm}]$,

$$[H'_{\mu'\mu}] = S_{m'} S_m [U^\dagger{}_{\mu'm'}][H_{m'm}][U_{m\mu}]\ , \quad \forall \mu', \mu\ , \tag{6.39}$$

where μ runs the same values as m does.

*In [25] the letter P has been used instead of η_1.

In their turn the subblocks in question may be considered as the matrices of operators which act onto fermionic degrees of freedom. Such operators have been represented in [25] as functions of the bare creation–destruction operators but fermion ones only. Let us denote the set of them by the symbol $\hat{f}$. Then, one can write, e.g.,

$$[U_{m'm}(\hat{f})]_{f'f} = \langle m'f' \mid U \mid mf \rangle. \tag{6.40}$$

Similarly we introduce the operators $H'_{m'm}(\hat{f})\,(H_{m'm}(\hat{f}))$ which correspond to $[H'_{m'm}]([H_{m'm}])$.

Under such convention the relation $H' = U^{\dagger}HU$ can be reduced to a hybrid form being expressed in terms of the functions of $\hat{f}$ and the matrix elements with respect to the meson states,

$$H'_{\mu'\mu}(\hat{f}) = S_{m'}S_m U^{\dagger}{}_{\mu'm'}(\hat{f})H_{m'm}(\hat{f})U_{m\mu}(\hat{f}). \tag{6.41}$$

Such representation of the operators turns out to be convenient when finding approximate solutions of Eq. (6.22) which can be rewritten as

$$\eta_2\{H_I + [H_F, \mathcal{A}] + [H_I, \mathcal{A}] - \mathcal{A}H_I\mathcal{A}\}\eta_1 = 0 \tag{6.42}$$

since $\eta_2 H_F \eta_1 = 0$.

We consider its approximate solution assuming

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)}, \mathcal{A}^{(0)} \sim g^0, \mathcal{A}^{(1)} \sim g^1. \tag{6.43}$$

The contribution $\mathcal{A}^{(0)}$ is absent since with interaction switched off ($g \to 0$) the Hamiltonian $H = H_F$ is already block-diagonal. In this connection, remind that $H_I = V + M_{\rm ren}$, where the mass counterterms $M_{\rm ren} = M_{\rm ferm} + M_{\rm mes}$ are determined by Eq. (2.7).

Such $\mathcal{A} = \mathcal{A}^{(1)}$ turns into zero only those terms of Eq. (6.42) which are of the g^1–order, i.e.,

$$\eta_2\left([H_F, \mathcal{A}] + V\right)\eta_1 = 0\ . \tag{6.44}$$

One can consider Eq. (6.44) as a relaxed form of Okubo's constraint $H'^{21} = 0$ (or Eq. (6.42)), which may be imposed instead of the latter.

Using the commutativity of H_F and η_1 let us rewrite Eq. (6.44) as

$$[H_F, \mathcal{A}] + \eta_2 V \eta_1 = 0\ . \tag{6.45}$$

This equation is of the same type as Eq. (2.21) for the operator R, viz., $[H_F, R] - V = 0$, whose solution is given in App. A. Therefore, taking into account formula (A.4) we find,

$$\mathcal{A} = \imath \lim_{\epsilon \to 0+} \int_0^{\infty} V^{21}(t) \exp(-\epsilon t) dt\ . \tag{6.46}$$

For interaction V linear in the meson field we have $A_{m0}(\hat{f}) = 0$ if the index m corresponds to two-, three-, ... meson states. Let us consider the nonzero subblock $A_{\mathbf{k}0}$, i.e., the subblock with elements $\langle \mathbf{k}f' \mid \mathcal{A} \mid 0f \rangle$. With the help of the representation (A.9) for the interaction V and relation

$$\langle a^\dagger(\mathbf{k})\Omega \mid a^\dagger(\mathbf{k}', t) \mid \Omega \rangle = \delta(\mathbf{k} - \mathbf{k}') \exp \imath\omega_{\mathbf{k}} t$$

we obtain

$$A_{\mathbf{k}0}(\hat{f}) = \imath \lim_{\epsilon \to 0+} \int_0^\infty V_{\mathbf{k}0}(\hat{f}(t)) \exp\left[\imath(\omega_{\mathbf{k}} + \imath\epsilon)t\right] dt \ , \tag{6.47}$$

with

$$V_{\mathbf{k}0}(\hat{f}(t)) = F^\dagger(t) V^{-\mathbf{k}} F(t) \ , \tag{6.48}$$

where $\hat{f}(t) = \exp\left[\imath H_{F\mathrm{ferm}}(\hat{f})t\right] \hat{f} \exp\left[-\imath H_{F\mathrm{ferm}}(\hat{f})t\right]$ is the subset of the fermion operators in the interaction picture. We imply the division $H_F = H_{F\mathrm{mes}}(\hat{m}) + H_{F\mathrm{ferm}}(\hat{f})$ into the mesonic and nucleonic parts (see Eq. (2.8)), $\hat{m}$ being the subset of the meson operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})\,\forall\mathbf{k}$. For notations F and $V^{-\mathbf{k}}$ see App. A.

At the same time the $(\mathbf{k}0)$ subblock of the operator $R = \mathcal{R} - \mathcal{R}^\dagger$ determined by Eqs. (A.4) and (A.9) is

$$R_{\mathbf{k}0}(\hat{f}) = -\imath \lim_{\epsilon \to 0+} \int_0^\infty V_{\mathbf{k}0}(\hat{f}(t)) \exp\left[\imath(\omega_{\mathbf{k}} + \imath\epsilon)t\right] dt. \tag{6.49}$$

Comparing Eqs. (6.47) and (6.49), we get the solution for $A_{\mathbf{k}0}$,

$$A_{\mathbf{k}0} = -R_{\mathbf{k}0} = \mathcal{R}^\dagger{}_{\mathbf{k}0} = F^\dagger R^{\mathbf{k}\dagger} F \tag{6.50}$$

in a compact form.

Here, we do not intend to write down explicit expressions for these subblocks in terms of the fermion creation (destruction) operators. The respective result for $A^{(1)}_{\mathbf{k}0}(\hat{f})$ with some extension to other Yukawa-type meson–nucleon couplings can be found in [25] (see Eqs. (17)–(18) therein).

After all, in order to derive an approximation to the operator H'^{11}, let us employ the expansion of $(1 + \mathcal{A}^\dagger\mathcal{A})^{-\frac{1}{2}}$ in the powers of $\mathcal{A}^\dagger\mathcal{A} \sim g^2$,

$$(1 + \mathcal{A}^\dagger\mathcal{A})^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\mathcal{A}^\dagger\mathcal{A}. \tag{6.51}$$

Then, we obtain from Eq. (6.23) the expression

$$H'^{11} = \eta_1\{H + \mathcal{A}^\dagger H + H\mathcal{A} + \ \frac{1}{2}\mathcal{A}^\dagger[H, \mathcal{A}] - \ \frac{1}{2}[H, \mathcal{A}^\dagger]\mathcal{A}\}\eta_1 + O(g^4) \ , \tag{6.52}$$

from which and condition (6.22) for $\mathcal{A}$ it follows that

$$H'^{11} = H_{F\mathrm{ferm}}(\hat{f})\eta_1 + M^{(2)}_{\mathrm{ferm}}(\hat{f})\eta_1 + \frac{1}{2}\eta_1\{\mathcal{A}^\dagger H + H\mathcal{A}\}\eta_1 + O(g^4)\ , \quad (6.53)$$

or neglecting the terms of order g^4,

$$H'^{11} = H_{F\mathrm{ferm}}(\hat{f})\eta_1 + M^{(2)}_{\mathrm{ferm}}(\hat{f})\eta_1 + \frac{1}{2}\eta_1\{\mathcal{A}^{(1)\dagger} V + V\mathcal{A}^{(1)}\}\eta_1 \qquad (6.54)$$

(cf. Eq. (15) of [25]). When deriving Eq. (6.53) we have used the equation

$$\eta_1 H \eta_1 = H_{F\mathrm{ferm}}(\hat{f})\eta_1 + M^{(2)}_{\mathrm{ferm}}(\hat{f})\eta_1 + O(g^4)\ ,$$

whereas the change from (6.53) to (6.54) is based upon the properties

$$\eta_2 H \eta_1 = \eta_2 M^{(2)}_{\mathrm{mes}}(\hat{m})\eta_1 + \eta_2 V \eta_1 + O(g^4),$$

$$\eta_1 H \eta_2 = \eta_1 M^{(2)}_{\mathrm{mes}}(\hat{m})\eta_2 + \eta_1 V \eta_2 + O(g^4),$$

and the fact that

$$\eta_1 M^{(2)}_{\mathrm{mes}}(\hat{m})\mathcal{A}\eta_1 = O(g^4)\ ,$$

$$\eta_1 V \mathcal{A}\eta_1 = \eta_1 V \mathcal{A}^{(1)}\eta_1 + O(g^4)\ ,$$

for V linear in the meson field.

The corresponding subblock H'_{11} can be expressed through the subblocks $A_{\mathbf{k}0}$ and $V_{\mathbf{k}0}$,

$$H'_{11} = H_{F\mathrm{ferm}}(\hat{f}) + M^{(2)}_{\mathrm{ferm}}(\hat{f}) + \frac{1}{2}\int\{A^\dagger{}_{0\mathbf{k}}(\hat{f})V_{\mathbf{k}0}(\hat{f}) + \mathrm{H.c.}\}d\mathbf{k} =$$

$$H_{F\mathrm{ferm}}(\hat{f}) + M^{(2)}_{\mathrm{ferm}}(\hat{f}) + \frac{1}{2}\int\{F^\dagger R^{\mathbf{k}} F \cdot F^\dagger V^{-\mathbf{k}} F + \mathrm{H.c.}\}d\mathbf{k}, \qquad (6.55)$$

where we have employed Eq. (6.50) and Eq. (6.48) at $t = 0$ and their H.c.

6.6. Clothing Procedure vs. Okubo Approach. Let us discuss in detail common and distinctive features of the two kinds of UT. Partly, we have concerned this subject in Subsec. 6.4 (see also App. C).

The Okubo and clothing approaches differ in their goals. The UT by Okubo is aimed at to nullify the nondiagonal blocks H'_{21} and H'_{12} of the transformed Hamiltonian. The clothing and akin approaches (see Sec. 5) require that some undesirable operator terms («bad» or «virtual») must be absent in the transformed Hamiltonian.

In the context it is worthwile to mention Nishijima's modification [9] of the Okubo idea. Instead of Eq. (6.8) he used the representation

$U = \ldots \exp(\imath S_2) \exp(\imath S_1)$, similar to that employed within the clothing approach. But Hermitian operators S_n ($n = 1, 2, 3, \ldots$) are to implement Okubo's requirement, viz., the operator S_n must remove K_{21} matrix elements of the g^n-order (instead of removing some bad terms $\sim g^n$).

No wonder different approaches give, in general, different resulting transformed Hamiltonians. Nevertheless, we want to show that both the approaches can give some coincident results (even for the realistic field model), being realized approximately (in a perturbative way). In particular, effective quasipotentials of the g^2-order for nucleon-nucleon interaction turn out to be the same.

Before comparing the resulting Hamiltonians we must remind that Okubo's H' (see Subsec. 6.5) does not coincide with the starting Hamiltonian H whereas the operator K of the clothing approach does. H' can be considered as a function of bare operators a while K depends on clothed ones α. However, we have shown in Subsec. 6.4 that H coincides with the operator $H'(\tilde{a})$ which depends upon new destruction–creation operators $\tilde{a} = UaU^\dagger$ in the same manner as H' depends on a. So, $H'(\tilde{a})$ and K represent the same operator and can be compared to each other. Let us recall, however, that $\tilde{a}$ may not coincide with clothed operators α, and H' and K may not be the same function of their arguments. In what follows we imply that H' means $H'(\tilde{a})$ and H'_{11} is a function of fermion destruction-creation operators $\tilde{f} = UfU^\dagger$.

Now, we can prove the statement: H'_{11} given by Eq. (6.55) coincides with the fermionic part of $K_F + M^{(2)}_{\rm ren} + \frac{1}{2}[R, V]$ (see Eq. (2.23)), i.e., with

$$K_{\rm ferm} = K_{F\rm ferm} + M^{(2)}_{\rm ferm} + (\text{fermionic part of } \frac{1}{2}[R, V]) \ . \tag{6.56}$$

More exactly, H'_{11} and $K_{\rm ferm}$ are the same function of their arguments (fermionic operators $\tilde{f}$ and f_c, respectively). Indeed, the last term in Eq. (6.56) is that part of $\frac{1}{2}[R, V] \sim g^2$ which depends only on fermionic operators. We calculate $\frac{1}{2}[R, V]$ in App.A. Its fermionic part (denoted below as FP) is contained in $\frac{1}{2}[\mathcal{R}, \mathcal{V}^\dagger] + \text{H.c.}$, for $[\mathcal{R}, \mathcal{V}^\dagger]$ see the last term in the r.h.s. of Eq. (A.14). We have

$$FP \equiv \frac{1}{2} \int [F^\dagger R^{\mathbf{k}} F \cdot F^\dagger V^{-\mathbf{k}} F + \text{H.c.}] d\mathbf{k} \ . \tag{6.57}$$

Comparing this expression with the r.h.s. of Eq. (6.55), we arrive to the above statement.

The fermion mass counterterm $M^{(2)}_{\rm ferm}$ in Eqs. (6.55) or (6.56) must cancel two-operator fermionic contributions which arise after normal ordering of FP (see Subsec. 2.5). One should note that there are no other such terms in $\frac{1}{2}[R, V]$. So, we may omit $M^{(2)}_{\rm ferm}$ from (6.55) or (6.56) provided FP is replaced by the normally ordered counterpart $:FP:$ of (6.57).

Let us mention that similar evaluation of $M^{(2)}_{\rm mes}$ (cf. Subsec. (2.5)) would take additional efforts within Okubo's approach (see Note at the end of this subsection).

Of course, we may explicitly express FP given by Eq. (6.57) through $b, d, b^\dagger$, and $d^\dagger$ opening the abbreviations accepted for $F^\dagger V^{\mathbf{k}} F$ and $F^\dagger R^{\mathbf{k}} R$ in App. A (we omit here the symbol tilde or the subscript c when handling with the fermion operators). Such representation shows that $:FP:$ contains «bad» terms of the kind $b^\dagger d^\dagger b^\dagger d^\dagger$, $b^\dagger d^\dagger b^\dagger b$ and $b^\dagger d^\dagger d^\dagger d$. As stressed in Subsec. 2.4, they should be eliminated from K via the clothing transformation W_4. Such terms are also unpleasant within Okubo's approach, viz., they prevent the no-fermion state Ω_f and one-fermion ones $b^\dagger\Omega_f$ and $d^\dagger\Omega_f$ to be H'_{11} eigenvectors along with two-fermion states of the kind (4.14).

In other words, the problem of finding H'_{11} eigenvectors is not essentially easier than that for the starting Hamiltonian. In the spirit of Okubo's approach the elimination of «bad» terms can be implemented by performing UT of H'_{11} such that the transformed sub-Hamiltonian $(H'_{11})' = U'^\dagger H'_{11} U'$ would not contain matrix elements corresponding to the processes $\Omega_f \to$ two pairs, $N \to N +$ pair, etc. The relevant projector η'_1 may project on states without pairs.

As argued in Sec. 4, these additional transformations W_4 or U' do not alter the remaining «good» four-fermionic pieces of $:FP:$ which are of the g^2-order (see, e.g., Eq. (4.18)).

So we obtain from H'_{11} or $K_{\rm ferm}$ the same expression $H_{F\rm ferm} + G:FP:$, where $G:FP:$ is a «good» part of $:FP:$. Using the notations of Sec. 4 (see Eqs. (4.1)–(4.3)), it may be written as $(K_{ZA})_{\rm ferm}$ or

$$K_{2\rm ferm} + K(NN \to NN) + K(\bar N\bar N \to \bar N\bar N) + K(N\bar N \to N\bar N)\ . \qquad (6.58)$$

The no-fermion, one-fermion and two-fermion deuteron-like eigenvectors of this operator give some approximations to the corresponding H eigenvectors.

Note. The zeroth approximation operator K_{ZA} considered in Sec. 4 enables us to find also approximations to H eigenvectors which describe one-meson states and meson–nucleon bound and scattering states. In the Okubo–Korchin–Shebeko approach exposed in Subsec. 6.5 one should use for this purpose the submatrix H'_{22}. However, the latter embodies, in general, undesirable matrix elements which correspond to the processes one-meson $\to$ other states (e.g., two meson + pair), etc. In order to simplify the finding of H eigenvectors under discussion one needs additional Okubo's transformation of the kind $(H'_{22})' = \tilde U^\dagger H'_{22} \tilde U$ such that $(H'_{22})'$ would have no undesirable matrix elements.

7. CONCLUSIONS

A considerable part of this work is devoted to development of the UT method using the so-called clothing procedure in RQFT. This procedure has two aspects.

On the one hand, we express with its aid a total Hamiltonian H for interacting fields in terms of the new operators which correspond to the creation (destruction) of clothed particles. The latter possess, by definition, the properties of observed (physical) particles and are as a matter of fact «quasiparticles» within our approach, if one draws a parallel with the method of canonical transformations in quantum theory (see, e.g., [55], §§52, 84). On the other hand, such representation for H, being implemented partially or perfectly, enables us to formulate an approach to solution of the H eigenvalue problem.

We have discussed in detail various UT of H in Sec. 6. In this connection, we distinguish two kinds of UT's destined for approximate determination of H eigenstates. Both may be deduced from the relation between the Hamiltonian H matrices with respect to two different sets of basis vectors.

The first kind may be written as $H' = U^{\dagger} H U$, where H is the input Hamiltonian which is subject to UT, the transformed operator H' being not equal to the input one. However, H eigenstates can be obtained from those of H' using U (see Subsec. 6.3).

The second kind is determined as $H = U^{\dagger} \tilde{H} U$ ($U = W^{\dagger}$). Now, the transformed Hamiltonian is the input one, but differently represented, while the operator $\tilde{H} \neq H$ is related to H in a simple way. These kinds of UT's are used in literature but authors sometimes overlook that H' eigenstates do not coincide with those of H.

Applying either of such UT's one may impose different constraints on the form of the transformed Hamiltonian that leads to the definite recipes for constructing the corresponding unitary operators. We have shown the resemblances and distinctions of some known applications of the UT method: the clothing procedure, the approach by Heitler and Sato, and Okubo's block-diagonalization (Secs. 5 and 6). One should note that in all applications the unitary operator U is determined approximately (with the exception of simple models).

Our consideration of the problem of bound and scattering states differs from akin approaches (cf., [9,35]) with the following distinctive feature. Our interactions between clothed particles (as an illustration, the πN and NN quasipotentials) are parts of a single operator K_{ZA} which can be regarded as a zeroth approximation to the total Hamiltonian. If the NN system is considered, then K_{ZA} generates the NN Hamiltonian, whereas for the πN system K_{ZA} gives rise to the πN Hamiltonian. This makes clear the relation of the eigenstates of such Hamiltonian to the eigenstates of the original field Hamiltonian: the former are some approximations to the latter.

One should point out that nonlocal properties of these quasipotentials have a relativistic origin. In this context, we would like to note a growing interest in similar relativistic effects in the modern theories of nuclear forces (see, e.g., [56,57] and refs. therein). We show how the mass renormalization program is

realized within the approach developed here. The transformed Hamiltonian when expressed in terms of clothed operators turns out to be dependent on renormalized (physical) masses and not bare ones. Some tricks shown in App. A can be useful in future calculations of the radiative corrections (renormalizations) to bare (trial) masses for field models with a cutoff in the momentum representation.

We prove that clothing UT's remove the undesired («bad») terms simultaneously from the Hamiltonian and the generators of Lorentz boosts. Our proof is valid for any RQFT model (see Sec. 3).

Our three-dimensional formalism is covariant in that sense that we give definite prescriptions for the transformation properties of the clothed states with respect to the Lorentz boosts.

With the help of a simple example, we demonstrate in App. B that the clothing transformation W of Sec. 2 may happen to be implemented not by a unitary operator in its usual mathematical sense. We argue that W ought to be understood as an element of some algebra (lacking any operator representation), being unitary in an algebraic sense.

At last, we show in App. C original tricks and results concerning a nonperturbative solution of Okubo's decoupling equation.

We believe that the concept of clothed particles and the approach exposed here can be applied to different areas of the nuclear physics: from the theory of nuclear structure to description of nuclear reactions including the processes with meson production.

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APPENDIX A

Three-Operator Clothing Transformation for the Yukawa Model. Four-Operator Interactions between Clothed Particles and Their Normal Ordering. The defining equations for the Yukawa model are given in Sec. 2. We use throughout this article notations of [38] assuming, in particular, $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$ ($\mu = 0,1,2,3$), $\hat{q} = q^\mu\gamma_\mu$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5^\dagger$ and $\gamma_5^2 = 1$.

A1. Three-operator clothing transformation $W = \exp R$, see Sec. 2.4, can be found by solving the equation $[R, H_{\rm F}] + V = 0$. The antihermitian operator R is assumed to be of the form $R = \mathcal{R} - \mathcal{R}^\dagger$ with $\mathcal{R}$ given by Eq. (2.22). The commutator $[R, H_{\rm F}]$ can be directly evaluated by using the commutation relations (2.5). Then we obtain the equations for the coefficients R_{ij} involved in $\mathcal{R}$, see

Eq. (2.22). Their solutions are

$$\begin{aligned} R^{\mathbf{k}}_{11}(\mathbf{p}'r';\mathbf{p}r) &= V^{\mathbf{k}}_{11}(\mathbf{p}'r';\mathbf{p}r)/(E_{\mathbf{p}'} - E_{\mathbf{p}} - \omega_{\mathbf{k}}) \ , \\ R^{\mathbf{k}}_{12}(\mathbf{p}'r';\mathbf{p}r) &= V^{\mathbf{k}}_{12}(\mathbf{p}'r';\mathbf{p}r)/(E_{\mathbf{p}'} + E_{\mathbf{p}} - \omega_{\mathbf{k}}) \ , \\ R^{\mathbf{k}}_{21}(\mathbf{p}'r';\mathbf{p}r) &= V^{\mathbf{k}}_{21}(\mathbf{p}'r';\mathbf{p}r)/(-E_{\mathbf{p}'} - E_{\mathbf{p}} - \omega_{\mathbf{k}}) \ , \\ R^{\mathbf{k}}_{22}(\mathbf{p}'r';\mathbf{p}r) &= V^{\mathbf{k}}_{22}(\mathbf{p}'r';\mathbf{p}r)/(-E_{\mathbf{p}'} + E_{\mathbf{p}} - \omega_{\mathbf{k}}) \ . \end{aligned} \tag{A.1}$$

Here we have used the notations

$$\left[\begin{array}{cc} V^{\mathbf{k}}_{11}(\mathbf{p}'r';\mathbf{p}r) & V^{\mathbf{k}}_{12}(\mathbf{p}'r';\mathbf{p}r) \\ V^{\mathbf{k}}_{21}(\mathbf{p}'r';\mathbf{p}r) & V^{\mathbf{k}}_{22}(\mathbf{p}'r';\mathbf{p}r) \end{array} \right] = i\frac{g}{(2\pi)^{3/2}}\frac{m}{\sqrt{2\omega_{\mathbf{k}}E_{\mathbf{p}'}E_{\mathbf{p}}}}\delta(\mathbf{p}+\mathbf{k}-\mathbf{p}')\times$$

$$\times \left[\begin{array}{cc} \bar{u}(\mathbf{p}',r')\gamma_5 u(\mathbf{p},r) & \bar{u}(\mathbf{p}',r')\gamma_5 v(-\mathbf{p},r) \\ \bar{v}(\mathbf{p}',r')\gamma_5 u(\mathbf{p},r) & \bar{v}(\mathbf{p}',r')\gamma_5 v(-\mathbf{p},r) \end{array} \right] \equiv V^{\mathbf{k}}(\mathbf{p}'r';\mathbf{p}r) \ . \tag{A.2}$$

Eqs. (A.1) have meaning if the denominators in their r.h.s. are not zero (note that $\mathbf{p}+\mathbf{k}-\mathbf{p}'=0$ according to (A.2)). One can show that this is the case under the condition

$$\mu < 2m \ . \tag{A.3}$$

The physical sense of this condition is discussed in Subsec. 2.4.

Alternatively, the solution R of Eq. (2.21) can be represented as

$$R = -i \lim_{\varepsilon\to 0+} \int_0^\infty V(t)\mathrm{e}^{-\varepsilon t}\mathrm{d}t \ , \tag{A.4}$$

where $V(t) = \exp(iH_{\mathrm{F}}t)V\exp(-iH_{\mathrm{F}}t)$ is the interaction operator in the Dirac picture. Obviously, $V(t)$ is determined by Eq. (2.16) where the operators $a_{\mathrm{c}}(\mathbf{k})$, $b_{\mathrm{c}}(\mathbf{p},r)$ and $d_{\mathrm{c}}(\mathbf{p},r)$ are replaced by $a_{\mathrm{c}}(\mathbf{k})\exp(-i\omega_{\mathbf{k}}t)$, $b_{\mathrm{c}}(\mathbf{p},r)\exp(-iE_{\mathbf{p}}t)$ and $d_{\mathrm{c}}(\mathbf{p},r)\exp(-iE_{\mathbf{p}}t)$, respectively. Therefore, the evaluation of R is reduced to integrals of the kind

$$\int_0^\infty \mathrm{e}^{i(x+i\varepsilon)t}\ \mathrm{d}t = \frac{i}{x+i\varepsilon} \ , \tag{A.5}$$

where x is any of the denominators in (A.1) and $\varepsilon > 0$ *. The limit $\varepsilon \to 0+$ in Eq. (A.4) exists, and it is finite if the inequality (A.3) takes place, i.e., if $x \neq 0$. This evaluation of R shows readily that the solution given by Eq. (A.4) coincides with that determined by Eqs. (A.1). Also, one can directly verify that (A.4) meets Eq. (2.21). In fact, we have

$$[R, H_{\mathrm{F}}] = -i \lim_{\varepsilon\to 0+} \int_0^\infty \mathrm{e}^{-\varepsilon t}[V(t), H_{\mathrm{F}}]\ \mathrm{d}t =$$

*According to the prescription given by Eq. (A.4) the positive parameter ε should be put equal to zero at the end of the calculations.

$$= \lim_{\varepsilon \to 0+} \int_0^\infty \mathrm{e}^{-\varepsilon t} \frac{\partial}{\partial t} V(t)\ \mathrm{d}t \ = \ -V \ . \tag{A.6}$$

The last equality follows (under the condition $\mu < 2m$) if one calculates $\frac{\partial}{\partial t} V(t)$ and then integrates with the help of (A.5). We use the form (A.4) in Sec. 3 when constructing the Lorentz boosts in terms of the clothed operators.

A2. Evaluation of $[R, V]$ is a tedious exercise that can be simplified with the aid of a more compact notation. Indeed, expressions (2.16) and (2.22) for V and R contain the identical operator structure, viz.,

$$b_{\mathrm{c}}^\dagger(\mathbf{p}', r') X_{11}^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) b_{\mathrm{c}}(\mathbf{p}, r) \ + \ b_{\mathrm{c}}^\dagger(\mathbf{p}', r') X_{12}^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) d_{\mathrm{c}}^\dagger(-\mathbf{p}, r) +$$
$$+ \ d_{\mathrm{c}}(-\mathbf{p}', r') X_{21}^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) b_{\mathrm{c}}(\mathbf{p}, r) \ + \ d_{\mathrm{c}}(-\mathbf{p}', r') X_{22}^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) d_{\mathrm{c}}^\dagger(-\mathbf{p}, r) \tag{A.7}$$

with definite c-number coefficients $X_{ij}^{\mathbf{k}}$ $(i, j = 1, 2)$.

Now, let us rewrite (A.7) as a matrix product

$$(b_{\mathrm{c}}^\dagger(\mathbf{p}', r'),\ d_{\mathrm{c}}(-\mathbf{p}', r')) \begin{bmatrix} X_{11}^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) & X_{12}^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) \\ X_{21}^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) & X_{22}^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) \end{bmatrix} \begin{pmatrix} b_{\mathrm{c}}(\mathbf{p}, r) \\ d_{\mathrm{c}}^\dagger(-\mathbf{p}, r) \end{pmatrix} \equiv$$
$$\equiv \ F^\dagger(\mathbf{p}', r') X^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) F(\mathbf{p}, r) \ , \tag{A.8}$$

where along with the 2×2 matrix $X^{\mathbf{k}}$ composed of the coefficients $X_{ij}^{\mathbf{k}}$ (cf. Eq. (A.2)) we have introduced the operator column F and row $F^\dagger$,

$$F(\mathbf{p}, r) = \begin{pmatrix} b_{\mathrm{c}}(\mathbf{p}, r) \\ d_{\mathrm{c}}^\dagger(-\mathbf{p}, r) \end{pmatrix} \ , \qquad F^\dagger(\mathbf{p}', r') = (b_{\mathrm{c}}^\dagger(\mathbf{p}', r'),\ d_{\mathrm{c}}(-\mathbf{p}', r')) \ .$$

The subsequent operations become even more concise after adopting the convention

$$\int \mathrm{d}\mathbf{p}' \int \mathrm{d}\mathbf{p} \sum_{r'r} F^\dagger(\mathbf{p}', r') X^{\mathbf{k}}(\mathbf{p}'r'; \mathbf{p}r) F(\mathbf{p}, r) \equiv F^\dagger X^{\mathbf{k}} F \ . \tag{A.8$'$}$$

Under these notations, Eqs. (2.16) and (2.22) look as

$$V = \int \mathrm{d}\mathbf{k} \ F^\dagger V^{\mathbf{k}} F \ [a_{\mathrm{c}}(\mathbf{k}) + a_{\mathrm{c}}^\dagger(\boldsymbol{-k})] =$$
$$= \int \mathrm{d}\mathbf{k} \ F^\dagger V^{\mathbf{k}} F a_{\mathrm{c}}(\mathbf{k}) \ + \ \mathrm{H.c.} \ \equiv \ \mathcal{V} + \mathcal{V}^\dagger \ , \tag{A.9}$$
$$R = \int \mathrm{d}\mathbf{k} \ F^\dagger R^{\mathbf{k}} F a_{\mathrm{c}}(\mathbf{k}) \ - \ \mathrm{H.c.} \ = \ \mathcal{R} - \mathcal{R}^\dagger \ ,$$

where H.c. means taking the Hermitian conjugate of the first terms. Note that $(V^{\mathbf{k}})^\dagger = V^{\mathbf{k}}$.

A.3. After these preliminaries, we have

$$[R,V]=[\mathcal{R}-\mathcal{R}^\dagger,V]=[\mathcal{R},V]+[V^\dagger,\mathcal{R}^\dagger]=[\mathcal{R},V]+\text{H.c.} \tag{A.10}$$

Keeping this in mind, it is sufficient to evaluate

$$[\mathcal{R},V]=[\mathcal{R},\mathcal{V}]+[\mathcal{R},\mathcal{V}^\dagger]\ . \tag{A.11}$$

By using the definitions from (A.9) and carrying out straightforward operator algebra, we obtain step by step

$$[\mathcal{R},\mathcal{V}]=\int \mathrm{d}\mathbf{k}_1\ \int \mathrm{d}\mathbf{k}_2\ [F^\dagger R^{\mathbf{k}_1}F,\ F^\dagger V^{\mathbf{k}_2}F]\ a_{\mathrm{c}}(\mathbf{k}_1)a_{\mathrm{c}}(\mathbf{k}_2)=$$

$$=\int \mathrm{d}\mathbf{k}_1\ \int \mathrm{d}\mathbf{k}_2\ F^\dagger[R^{\mathbf{k}_1},V^{\mathbf{k}_2}]Fa_{\mathrm{c}}(\mathbf{k}_1)a_{\mathrm{c}}(\mathbf{k}_2)\ , \tag{A.12}$$

where in accordance with Eq. (A.8′)

$$F^\dagger[R^{\mathbf{k}_1},V^{\mathbf{k}_2}]F=\int \mathrm{d}\mathbf{p}'\ \int \mathrm{d}\mathbf{p}\ \sum_{r'r}F^\dagger(\mathbf{p}',r')[R^{\mathbf{k}_1},V^{\mathbf{k}_2}](\mathbf{p}'r';\mathbf{p}r)\ F(\mathbf{p},r)\ ,$$

and it is implied that

$$[R^{\mathbf{k}_1},V^{\mathbf{k}_2}](\mathbf{p}'r';\mathbf{p}r)=$$

$$=\int \mathrm{d}\mathbf{q}\ \sum_s[R^{\mathbf{k}_1}(\mathbf{p}'r';\mathbf{q}s)V^{\mathbf{k}_2}(\mathbf{q}s;\mathbf{p}r)-V^{\mathbf{k}_2}(\mathbf{p}'r';\mathbf{q}s)R^{\mathbf{k}_1}(\mathbf{q}s;\mathbf{p}r)]\ . \tag{A.13}$$

In the above calculations it has been convenient to employ the identity

$$[AB,CD]=A\{B,C\}D-\{A,C\}BD-C\{D,A\}B+CA\{D,B\}$$

for four operators A, B, C, and D.

Further, after applying another useful relation

$$[AB,CD]=A[B,C]D+[A,C]DB+AC[B,D]+C[A,D]B\ ,$$

a similar derivation for $[\mathcal{R},\mathcal{V}^\dagger]$ results in

$$[\mathcal{R},\mathcal{V}^\dagger]=\int \mathrm{d}\mathbf{k}_1\ \int \mathrm{d}\mathbf{k}_2\ \{F^\dagger[R^{\mathbf{k}_1},V^{-\mathbf{k}_2}]Fa^\dagger_{\mathrm{c}}(\mathbf{k}_2)a_{\mathrm{c}}(\mathbf{k}_1)+$$

$$+\delta(\mathbf{k}_1-\mathbf{k}_2)F^\dagger R^{\mathbf{k}_1}F\cdot F^\dagger V^{-\mathbf{k}_2}F\}\ . \tag{A.14}$$

Now, we see that the g^2-order commutator $[R,V]$ brings in the total Hamiltonian K, see Eq. (2.23), the interaction terms which describe the following real physical processes:

a) the $\pi\pi \to f\bar{f}$ creation and the $f\bar{f} \to \pi\pi$ annihilation from $[\mathcal{R}, \mathcal{V}]$ (see Eq. (A.12)) and its H.c.;

b) the $\pi f \to \pi f$, $\pi\bar{f} \to \pi\bar{f}$, $ff \to ff$, $f\bar{f} \to f\bar{f}$ and $\bar{f}\bar{f} \to \bar{f}\bar{f}$ scatterings from $[\mathcal{R}, \mathcal{V}^\dagger]$ and its H.c. (see (A.14)).

In addition to these contributions, $[R, V]$ contains interactions, which have nonvanishing matrix elements between the vacuum Ω and two-particle states (e.g., those responsible for the virtual process $\Omega \to \pi\pi$), and between Ω and four-particle states (e.g., for transitions $\Omega \to \pi\pi f\bar{f}$). There are also interactions responsible for the transitions: one particle $\to$ three particles and one particle $\to$ one particle. Except the latter, all these terms are «bad», i.e., they hinder Ω and one-particle states to be K eigenstates.

For example, let us consider the term $d_{\rm c}d_{\rm c}^\dagger a_{\rm c}a_{\rm c}$ which enters in $[\mathcal{R}, \mathcal{V}]$. It has nonzero matrix element $\langle\Omega \mid d_{\rm c}d_{\rm c}^\dagger a_{\rm c}a_{\rm c} \mid \pi\pi\rangle$ that becomes evident after normal ordering,

$$d_{\rm c}(-\mathbf{p}', r')d_{\rm c}^\dagger(-\mathbf{p}, r)a_{\rm c}(\mathbf{k}_1)a_{\rm c}(\mathbf{k}_2) =$$

$$= -d_{\rm c}^\dagger(-\mathbf{p}, r)d_{\rm c}(-\mathbf{p}', r')a_{\rm c}(\mathbf{k}_1)a_{\rm c}(\mathbf{k}_2) + \delta_{r'r}\delta(\mathbf{p}' - \mathbf{p})\ a_{\rm c}(\mathbf{k}_1)a_{\rm c}(\mathbf{k}_2)\ .$$

This illustration shows that normal ordering is a constructive tool in the framework of our clothing procedure. As in many applications of the method of second quantization (e.g., in field theories of the evolution operator or the S-matrix), this operation enables us to classify the separate contributions to the original Hamiltonian at every stage of the clothing procedure.

A4. Now, we shall discuss in detail the origin of two-operator meson terms which stem from the commutator $[R, V]$. They are essential elements in our treatment of the particle mass renormalization (see Sec. 2.5). These terms appear after reshuffling the operators of the expressions $F^\dagger[R^{\mathbf{k}_1}, V^{\mathbf{k}_2}]Fa_{\rm c}(\mathbf{k}_1)a_{\rm c}(\mathbf{k}_2)$ (see Eq. (A.12)) and $F^\dagger[R^{\mathbf{k}_1}, V^{-\mathbf{k}_2}]Fa_{\rm c}^\dagger(\mathbf{k}_2)a_{\rm c}(\mathbf{k}_1)$ (see Eq. (A.14)) into normal order.

In the first case, it touches upon the terms of the $dd^\dagger$-kind and leads to

$$F^\dagger[R^{\mathbf{k}_1}, V^{\mathbf{k}_2}]F =: F^\dagger[R^{\mathbf{k}_1}, V^{\mathbf{k}_2}]F : + \mathrm{Tr}[R^{\mathbf{k}_1}, V^{\mathbf{k}_2}]_{22}\ , \tag{A.15}$$

where

$$\mathrm{Tr}[R^{\mathbf{k}_1}, V^{\mathbf{k}_2}]_{22} \equiv \int \mathrm{d}\mathbf{p}\ \sum_r [R^{\mathbf{k}_1}, V^{\mathbf{k}_2}]_{22}(\mathbf{p}r; \mathbf{p}r)\ , \tag{A.16}$$

$[R^{\mathbf{k}_1}, V^{\mathbf{k}_2}]_{22}(\mathbf{p}'r'; \mathbf{p}r) \equiv$

$$\equiv \int \mathrm{d}\mathbf{q}\ \sum_j \sum_s [R_{2j}^{\mathbf{k}_1}(\mathbf{p}'r'; \mathbf{q}s)V_{j2}^{\mathbf{k}_2}(\mathbf{q}s; \mathbf{p}r) - V_{2j}^{\mathbf{k}_2}(\mathbf{p}'r'; \mathbf{q}s)R_{j2}^{\mathbf{k}_1}(\mathbf{q}s; \mathbf{p}r)\ ]\ . \tag{A.17}$$

The trace Tr is evaluated using the properties of the γ-matrices and Dirac spinors. We find with the solutions (A.1)

$$\mathrm{Tr}[R^{\mathbf{k}_1}, V^{\mathbf{k}_2}]_{22} = 2\frac{t_{\mathbf{k}_1}}{\omega_{\mathbf{k}_1}}\delta(\mathbf{k}_1 + \mathbf{k}_2) \ , \tag{A.18}$$

$$t_{\mathbf{k}} = \frac{g^2}{4(2\pi)^3}\int \mathrm{d}\mathbf{p} \ \frac{m^2}{E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}}}\times$$
$$\times\left\{\frac{\mathrm{Sp}\,[\Lambda_-(-\mathbf{p})\Lambda_-(\mathbf{p}-\mathbf{k})]}{E_{\mathbf{p}}+\omega_{\mathbf{k}}+E_{\mathbf{p}-\mathbf{k}}} + \frac{\mathrm{Sp}\,[\Lambda_-(-\mathbf{p})\Lambda_+(\mathbf{p}+\mathbf{k})]}{E_{\mathbf{p}}+\omega_{\mathbf{k}}-E_{\mathbf{p}-\mathbf{k}}}+\right.$$
$$\left.+\ \frac{\mathrm{Sp}\,[\Lambda_-(-\mathbf{p})\Lambda_-(\mathbf{p}-\mathbf{k})]}{E_{\mathbf{p}}-\omega_{\mathbf{k}}+E_{\mathbf{p}-\mathbf{k}}} + \frac{\mathrm{Sp}\,[\Lambda_-(-\mathbf{p})\Lambda_+(-\mathbf{p}+\mathbf{k})]}{E_{\mathbf{p}}-\omega_{\mathbf{k}}-E_{\mathbf{p}-\mathbf{k}}}\right\} \tag{A.19}$$

with the notations $\Lambda_+(\Lambda_-)$ for the projection operators onto the nucleon positive (negative)-energy states:

$$\Lambda_\pm(\mathbf{q}) = \frac{\pm q + m}{2m} \ .$$

While deriving Eq. (A.19), we have taken into account that

$$\sum_s \bar{v}(\mathbf{p}s)Ov(\mathbf{p}s) = -\mathrm{Sp}\{\Lambda_-(\mathbf{p})O\} \ ,$$

where O is a combination of γ-matrices.

Of course, one can collect the similar terms with the same numerators in the r.h.s. of Eq. (A.19) (e.g., the first term with the third one inside the curly brackets). However, we prefer other continuation that enables us to get immediately an explicitly covariant form of $t_{\mathbf{k}}$.

First of all, we find

$$\mathrm{Sp}\,[\Lambda_-(-\mathbf{p})\Lambda_-(\mathbf{p}-\mathbf{k})] = \frac{E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}} + \mathbf{p}\cdot(\mathbf{p}-\mathbf{k}) + m^2}{m^2} \ ,$$

$$\mathrm{Sp}\,[\Lambda_-(-\mathbf{p})\Lambda_+(-\mathbf{p}+\mathbf{k})] = \frac{E_{\mathbf{p}}E_{\mathbf{p}-\mathbf{k}} - \mathbf{p}\cdot(\mathbf{p}-\mathbf{k}) - m^2}{m^2} \ .$$

Substituting these expressions into (A.19) and uniting therein the first term with the second one and the third term with the fourth one it can be shown that

$$t_{\mathbf{k}} = \frac{g^2}{2(2\pi)^3}\int \frac{\mathrm{d}\mathbf{p}}{E_{\mathbf{p}}} \ \left\{\frac{p_-k}{\mu^2+2p_-k} + \frac{pk}{\mu^2-2pk}\right\}, \tag{A.20a}$$

or

$$t_{\mathbf{k}} = \frac{g^2}{2(2\pi)^3}\int \frac{\mathrm{d}\mathbf{p}}{E_{\mathbf{p}}} \ \left\{1 + \frac{\mu^4}{4(pk)^2-\mu^4}\right\} \tag{A.20b}$$

with $p_- = (E_{\mathbf{p}}, -\mathbf{p})$, $p = (E_{\mathbf{p}}, \mathbf{p})$ and $k = (\omega_{\mathbf{k}}, \mathbf{k})$.

Since this integral is a Lorentz–scalar, one can write

$$t_{\mathbf{k}} = t_{\mathbf{k}=0} = \frac{g^2}{2(2\pi)^3} \int \frac{\mathrm{d}\mathbf{p}}{E_{\mathbf{p}}} \left\{ 1 + \frac{\mu^2}{4E_{\mathbf{p}}^2 - \mu^2} \right\} \tag{A.21}$$

or

$$t_{\mathbf{k}} = \frac{g^2}{4\pi^2} \left\{ I_1 + \frac{\mu^2}{4} \left[I_2 - \frac{\sqrt{4m^2 - \mu^2}}{\mu} \arctan \frac{\mu}{\sqrt{4m^2 - \mu^2}} \right] \right\} , \tag{A.22}$$

where the integrals

$$I_1 = \int_0^\infty \frac{x^2}{\sqrt{x^2 + m^2}} \, \mathrm{d}x$$

and

$$I_2 = \int_0^\infty \frac{\mathrm{d}x}{\sqrt{x^2 + m^2}}$$

are, respectively, quadratically and logarithmically divergent.

Now, the resulting contribution to $[\mathcal{R}, \mathcal{V}]$ which is of the $a_{\mathrm{c}} a_{\mathrm{c}}$-kind can be written as

$$2 \int \mathrm{d}\mathbf{k}_1 \int \mathrm{d}\mathbf{k}_2 \, \delta(\mathbf{k}_1 + \mathbf{k}_2) \frac{t_{\mathbf{k}_1}}{\omega_{\mathbf{k}_1}} a_{\mathrm{c}}(\mathbf{k}_1) a_{\mathrm{c}}(\mathbf{k}_2) = 2 \int \mathrm{d}\mathbf{k} \, \frac{t_{\mathbf{k}}}{\omega_{\mathbf{k}}} a_{\mathrm{c}}(\mathbf{k}) a_{\mathrm{c}}(-\mathbf{k}) \, . \tag{A.23}$$

The commutator $[R, V]$ includes also the Hermitian conjugate of $[\mathcal{R}, \mathcal{V}]$. Therefore, we obtain from $[\mathcal{R}, \mathcal{V}] + \mathrm{H.c.}$ the following "bad" two-operator meson contribution:

$$2 \int \mathrm{d}\mathbf{k} \, \frac{t_{\mathbf{k}}}{\omega_{\mathbf{k}}} \{ a_{\mathrm{c}}(\mathbf{k}) a_{\mathrm{c}}(-\mathbf{k}) + a_{\mathrm{c}}^\dagger(\mathbf{k}) a_{\mathrm{c}}^\dagger(-\mathbf{k}) \} \, . \tag{A.24}$$

In the case of $F^\dagger [R^{\mathbf{k}_1}, V^{-\mathbf{k}_2}] F a_{\mathrm{c}}^\dagger(\mathbf{k}_2) a_{\mathrm{c}}(\mathbf{k}_1)$ (see the beginning of this subsection) after normal ordering of the $dd^\dagger$-kind terms one has to deal with $\mathrm{Tr}[R^{\mathbf{k}_1}, V^{-\mathbf{k}_2}]_{22}$ that differs from Eq. (A.16) only by the replacement $\mathbf{k}_2 \to -\mathbf{k}_2$. Therefore, we obtain from $[\mathcal{R}, \mathcal{V}^\dagger]$ the following expression bilinear in the meson operators:

$$2 \int \mathrm{d}\mathbf{k} \, \frac{t_{\mathbf{k}}}{\omega_{\mathbf{k}}} a_{\mathrm{c}}^\dagger(\mathbf{k}) a_{\mathrm{c}}(\mathbf{k}) \, . \tag{A.25}$$

The same expression stems from the H.c. of $[\mathcal{R}, \mathcal{V}^\dagger]$.

Finally, uniting all these results one can write the entire contribution from $\frac{1}{2}[R, V]$ to $K(\alpha)$, which is bilinear in the meson operators,

$$\int \mathrm{d}\mathbf{k} \, \frac{t_{\mathbf{k}}}{\omega_{\mathbf{k}}} \{ 2 a_{\mathrm{c}}^\dagger(\mathbf{k}) a_{\mathrm{c}}(\mathbf{k}) + a_{\mathrm{c}}(\mathbf{k}) a_{\mathrm{c}}(-\mathbf{k}) + a_{\mathrm{c}}^\dagger(\mathbf{k}) a_{\mathrm{c}}^\dagger(-\mathbf{k}) \} \, . \tag{A.26}$$

APPENDIX B

Mathematical Aspects of the Clothing UT. By way of a simple example we shall show that W used in the framework of «clothing» approach (see Sec. 2) may happen to be not a unitary operator in the usual sense: unitary operator transforms vectors of a Hilbert space into vectors of the same space, the scalar products being conserved. We argue that W need not be such an operator, viz., the clothing program can be described using an algebraic language as if W is an element of some algebra, being unitary in an algebraic sense. Our example shows that such W can have a representation by an operator that transforms vectors of a Hilbert space $\mathcal{H}_0$ into vectors of another Hilbert space $\mathcal{H}$ which is orthogonal to $\mathcal{H}_0$. In general, the operator representation of W turns out to be unnecessary because W is not used in calculations of probability amplitudes, expectation values and other quantities which have physical interpretation. We note that «clothing» allows us to choose a proper Hilbert space for field model with the total Hamiltonian $H = H_0 + H_I$. This space usually is different from the space spanned on H_0 eigenvectors.

B1. The clothing program can explicitly be carried out in the following model. The scalar (meson) field $\phi(x)$ interacts with a fixed (external) source, the Hamiltonian being $H = H_0^{\mathrm{mes}} + g\int \phi(\mathbf{x})f(\mathbf{x})d\mathbf{x}$, where H_0^{mes} is the free meson Hamiltonian (see Eq.(2.1)). The Hamiltonian can be represented as

$$H = \int \omega_{\mathbf{k}}^0 a^\dagger(\mathbf{k})a(\mathbf{k})d\mathbf{k} + \int \omega_{\mathbf{k}}^0[v(\mathbf{k})\, a(\mathbf{k}) + v^*(\mathbf{k})\, a^\dagger(\mathbf{k})]d\mathbf{k} \tag{B.1}$$

with

$$v(\mathbf{k}) = \frac{g}{\sqrt{2(2\pi\omega_{\mathbf{k}}^0)^3}}\tilde{f}(\mathbf{k}),$$

where $\omega_{\mathbf{k}}^0 = \sqrt{\mathbf{k}^2 + \mu_0^2}$. The Fourier transform $\tilde{f}(\mathbf{k})$ of the source function $f(\mathbf{x})$ is a constant if the source is point-like: $f(\mathbf{x}) \sim \delta(\mathbf{x})$.

In accordance with the recipe (A.4) we find the corresponding clothing transformation $W = \exp R \equiv W(a_c) = W(a)$ with its generator

$$R = \int v(\mathbf{k})[a^\dagger(\mathbf{k}) - a(\mathbf{k})]d\mathbf{k}. \tag{B.2}$$

When deriving this expression, it is convenient to exploit the relation

$$\mathrm{e}^{\imath H_0^{\mathrm{mes}}t}a(\mathbf{k})\mathrm{e}^{-\imath H_0^{\mathrm{mes}}t} = \mathrm{e}^{-\imath\omega_{\mathbf{k}}^0 t}a(\mathbf{k}).$$

For brevity, we suppose that the form factor $v(\mathbf{k})$ is real ($v^*(\mathbf{k}) = v(\mathbf{k})$).

Further, one can verify that

$$a(\mathbf{k}) = Wa_c(\mathbf{k})W^\dagger = a_c(\mathbf{k}) - v(\mathbf{k}) \tag{B.3}$$

and

$$H = K(a_c) = \int \omega^0_{\mathbf{k}} a^{\dagger}{}_c(\mathbf{k}) a_c(\mathbf{k}) - L \ , \tag{B.4}$$

where $L = \int v(\mathbf{k})^2 d\mathbf{k}$ is the c-number that shifts the H spectrum.

Thus, this transformation W not only removes from H the «bad» terms (linear in a and $a^{\dagger}$) but it reduces the primary eigenvalue problem to a very simple one. In our model $K(a_c)$ does not contain interaction terms: clothed particles (mesons) turn out to be free.

Now, we are interested in the clothed no-particle state, i.e., the vector Ω such that

$$a_c(\mathbf{k})\Omega = 0, \quad \forall \mathbf{k} \tag{B.5}$$

with

$$\langle \Omega \mid \Omega \rangle = 1.$$

One can show that the vector $W^{\dagger}(a)\Omega_0$ obeys the condition (B.5). In fact,

$$a_c(\mathbf{k}) W^{\dagger}(a)\Omega_0 = W^{\dagger} a(\mathbf{k})\Omega_0 = 0. \tag{B.6}$$

So, one can put $\Omega = W^{\dagger}(a)\Omega_0$, i.e.,

$$\Omega = \exp\{-\int v(\mathbf{k})\,[a^{\dagger}(\mathbf{k}) - a(\mathbf{k})] d\mathbf{k}\}\Omega_0 =$$

$$= \exp(-L/2) \exp\{-\int v(\mathbf{k}) a^{\dagger}(\mathbf{k}) d\mathbf{k}\}\Omega_0. \tag{B.7}$$

Here, we have used the Hausdorff–Weyl formula,

$$\mathrm{e}^{A+B} = \mathrm{e}^{A}\mathrm{e}^{B}\mathrm{e}^{-1/2[A,B]} \tag{B.8}$$

for the two operators A and B such that $[A, B]$ commutes with A and B.

Equation (B.7) represents the «clothed» vacuum Ω as the superposition of «bare» states

$$\Omega_0, \quad a^{\dagger}(\mathbf{k}_1)\Omega_0, \quad a^{\dagger}(\mathbf{k}_1)\, a^{\dagger}(\mathbf{k}_2)\Omega_0, \ldots \tag{B.9}$$

Expansions similar to (B.7) can be written for «clothed» one-particle state $a^{\dagger}{}_c(\mathbf{k})\Omega$ as well as for all vectors of the kind

$$W^{\dagger} a^{\dagger}(\mathbf{k}_1) \ldots a^{\dagger}(\mathbf{k}_n)\Omega_0 = a^{\dagger}{}_c(\mathbf{k}_1) \ldots a^{\dagger}{}_c(\mathbf{k}_n) W^{\dagger}\Omega_0 \ , \tag{B.10}$$

each of them being the H eigenvector.

The states (B.9) are the H_0 eigenvectors and they form the basis of the Fock (Hilbert) space $\mathcal{H}_0$. The set

$$\Omega, \quad a^{\dagger}{}_c(\mathbf{k}_1)\Omega, \quad a^{\dagger}{}_c(\mathbf{k}_1)\, a^{\dagger}{}_c(\mathbf{k}_2)\Omega, \ldots \tag{B.11}$$

of the H eigenvectors is the basis of the Hilbert space which we shall call $\mathcal{H}$. We see that in our model with the cutoff function $v(\mathbf{k})$ which decreases rapidly enough to yield a finite normalization factor $\exp(-L/2)$, the space $\mathcal{H}$ can be spanned onto the vectors (B.9).

It is not the case in the model with a «soft» form factor $v(\mathbf{k})$. Indeed, if $L = \infty$ (this occurs, e.g., for the point-like source), then we obtain the zero values for all $W^\dagger\Omega_0$ projections on the vectors (B.9). Moreover, all vectors (B.10) are orthogonal to $\mathcal{H}_0$ if $L = \infty$. We may conclude that all vectors (B.11) are zero if $\mathcal{H}_0$ is assumed to be the complete space of states. The operator $W^\dagger$ transforms all $\mathcal{H}_0$ vectors into the zero one and therefore cannot be called the unitary operator because the latter, by definition, must transform vectors from $\mathcal{H}_0$ into vectors from $\mathcal{H}_0$ and conserve the scalar product in the space.

Note that in the case with $L \to \infty$ there are different ways for calculating the scalar product $\langle\Omega \mid \Omega\rangle = \langle W^\dagger\Omega_0 \mid W^\dagger\Omega_0\rangle$. On the one hand, putting $L = \infty$ in Eq. (B.7), we obtain $W^\dagger\Omega_0 = 0$ once $\mathcal{H}_0$ is complete. Therefore, $\langle W^\dagger\Omega_0 \mid W^\dagger\Omega_0\rangle = 0$. On the other hand, calculating at first $\langle\Omega \mid \Omega\rangle$ at finite L, we obtain unity for any L, i.e., such limit of $\langle W^\dagger\Omega_0 \mid W^\dagger\Omega_0\rangle$ as $L \to \infty$ is unity*.

B2. So, in the case $L = \infty$ the standard approach that is relied upon the initial Hilbert space $\mathcal{H}_0$ does not allow one to find H eigenstates (B.11). The situation was clarified by Van Hove [49]. He considered H as an operator in the space constructed as the infinite product of the Hilbert spaces for the oscillators $\omega_k a^\dagger(k)\, a(k)$ (Van Hove assumes that k runs discrete values). This extended space is not a Hilbertian one, but it can be decomposed into a direct sum of mutually orthogonal Hilbert spaces, $\mathcal{H}_0$ being one of them (e.g., see [50–52]).

By using this space Van Hove has proved that the H eigenvector which belongs to the least H eigenvalue (it coincides with the no-particle state Ω because of Eq. (B.4)) has unit norm and is orthogonal to $\mathcal{H}_0$. Moreover, all H eigenvectors (B.11) are orthogonal to $\mathcal{H}_0$. The vectors (B.11) form the basis of another Hilbert space which is orthogonal to $\mathcal{H}_0$. We have called it $\mathcal{H}$ above.

Of course, the model under discussion is too simplified and the resulting theory is equivalent to the free one, viz., H contains no interaction terms after «clothing» (see Eq. (B.4)). But it enables us to suspect that in nontrivial cases the total Hamiltonian eigenvectors may happen to be orthogonal to the initial Hilbert space $\mathcal{H}_0$, i.e., «bare» states space or space of the eigenvectors of the free part H_0 of the total Hamiltonian. On this ground one may cast doubt about validity of the usual quantum postulate that H as well as other observables can be

*In the context, we find in [15] (Ch. XII) that $\langle\Omega \mid \Omega\rangle = \infty$ at $L = \infty$. It follows from the supposition made therein that the projection $\Phi^{(0)} = \langle\Omega_0 \mid \Omega\rangle$ is not zero at $L = \infty$. However, this supposition is wrong (see Eq. (B.7) and Van Hove paper [49]).

defined initially as operators in $\mathcal{H}_0$. One can also anticipate that the «clothing» transformation may not be unitary operator in $\mathcal{H}_0$.

B3. In order to avoid the troubles described above we suggest the following algebraic approach to the quantum field theory. We consider all the operators occurred in quantum field theory as elements of an algebra, devoid of operator representation. This means that *ab initio* we do not introduce the notion of vectors, describing states of the physical system. In this algebra besides the addition and multiplication of elements an operation of involution † is defined, which corresponds to the Hermitian conjugation in the operator language (see, e.g., [53] (§1.5) and [54] (Ch. III)). The algebra contains $A^\dagger$ along with the element A. All elements of the algebra (in particular, the Hamiltonian) can be expressed in terms of some basic algebraic elements. In the case of Yukawa model the latter are

$$a(\mathbf{k}), a^\dagger(\mathbf{k}), b(\mathbf{p}, r), b^\dagger(\mathbf{p}, r), d(\mathbf{p}, r), d^\dagger(\mathbf{p}, r) \quad \forall \mathbf{k}, \mathbf{p}, r. \tag{B.12}$$

The multiplication operation is noncommutative: AB may not be equal to BA. The commutators $[A, B]$ for basic elements are assumed to be known (see, e.g., Eq. (2.3)), then the commutators of any two elements can be calculated.

The «clothing» program can be formulated and realized using this algebraic language. For example, the requirements i) and ii) from Sec. 2 can be replaced by equivalent ones: the Hamiltonian H when expressed in terms of new basic elements α_p (instead of the starting elements denoted as a_p in Sec. 2) must not contain «bad» terms (see Subsec. 2.4). The elements α_p and a_p are connected by the isomorphic transformation $\alpha_p = W^\dagger a_p W$, W being a unitary element of the algebra, i.e., such that $W^\dagger W = W W^\dagger = 1$. Let us stress that $W^\dagger a_p W$ is calculated using purely algebraic means, namely commutation relations and Eq. (2.15).

Of course, our theory must provide numbers, which can be compared with experimental data (cross sections, expectation values, etc.). In quantum mechanics this is accomplished by means of the realization of the above algebraic elements as operators in a vector space. We define the space as follows.

After «clothing» we introduce the additional notion of a state Ω (cyclic state) such that

$$a_c(\mathbf{k})\Omega = b_c(\mathbf{p}, r)\Omega = d_c(\mathbf{p}, r)\Omega = 0 \quad \forall \mathbf{k}, \mathbf{p}, r.$$

This state coincides with the H eigenstate corresponding to the least H eigenvalue. We assume that the observable no-particle state is described by Ω. By assumption observable one-particle states are described by the states

$$a^\dagger{}_c(\mathbf{k})\Omega, \quad b^\dagger{}_c(\mathbf{p}, r)\Omega, \quad d^\dagger{}_c(\mathbf{p}, r)\Omega.$$

The vectors

$$a^\dagger{}_c(\mathbf{k}_1) \dots a^\dagger{}_c(\mathbf{k}_n)\Omega, \quad b^\dagger{}_c(\mathbf{p}_1, r_1) \dots b^\dagger{}_c(\mathbf{p}_n, r_n)\Omega, \dots \quad (n \geq 2)$$

can be chosen as the remaining basic vectors of our Hilbert space $\mathcal{H}$. Using the vectors one can calculate quantities of physical interest in the usual manner.

One may say that one of the goals of the «clothing» is to select a Hilbert space which would be suitable for the given field Hamiltonian H. The first-quantized quantum mechanics uses only one universal Hilbert space for different Hamiltonians, the second-quantized theory needs distinct spaces for different interactions (e.g., for different values of the coupling constant g).

The approach described above needs not initial space $\mathcal{H}_0$. We need not to consider W as an operator in a space. Algebraically W is defined as a unitary element of the described algebra. Besides W the algebra has other elements which need not operator realization, the free part H_0 of the total Hamiltonian being the example.

APPENDIX C

The UT Method in Scalar Field Model. In order to illustrate the key points of clothing and Okubo's approaches let us consider another exactly solvable model [4, 15] in which a neutral scalar field is coupled with spinless fermions whose energy is independent of momentum. The model Hamiltonian is $H = H_0 + V$,

$$H_0 = m_0\, B(0) + \int \omega_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k}) d\mathbf{k}, \quad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu^2}\ , \tag{C.1}$$

$$V = g \int \omega_{\mathbf{k}} \left[B(\mathbf{k})\, a(\mathbf{k}) + \text{H.c.}\right]\ h(\mathbf{k}^2) \mathrm{d}\mathbf{k}, \quad h(\mathbf{k}^2) = \frac{f(\mathbf{k}^2)}{\sqrt{2(2\pi\omega_{\mathbf{k}})^3}} \tag{C.2}$$

with

$$B(\mathbf{k}) \equiv \int b^\dagger(\mathbf{p}+\mathbf{k})\, b(\mathbf{p}) \mathrm{d}\mathbf{p} = B^\dagger(-\mathbf{k})\ ,$$

where $a(\mathbf{k})$ and $b(\mathbf{p})$ are the destruction operators for bosons and fermions, respectively, which meet the usual commutation rules (cf. Eqs. (2.5)):

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}')\ , \tag{C.3a}$$

$$\{b(\mathbf{p}), b^\dagger(\mathbf{p}')\} = \delta(\mathbf{p} - \mathbf{p}')\ . \tag{C.3b}$$

The translational invariance of the Hamiltonian provides the momentum conservation. The cut-off factor $f(x)$ is assumed to fall off rapidly enough for large x to make finite all the integrals that occur in the theory.

C1. Again the clothing transformation of this Hamiltonian can be found in a closed form (cf. App. B). Indeed, calculating the respective integral (A.4) and noticing that

$$[B(\mathbf{k}), B(\mathbf{k}')] = 0 \quad \forall\ \ \mathbf{k}\ ,\ \mathbf{k}'\ , \tag{C.4}$$

we get

$$R = -g \int [B(\mathbf{k})a(\mathbf{k}) - \text{H.c.}]\ h(\mathbf{k}^2)\ \mathrm{d}\mathbf{k} \equiv g(X^\dagger - X) \tag{C.5}$$

with

$$X = \int h(\mathbf{k}^2) B(\mathbf{k}) a(\mathbf{k}) \mathrm{d}\mathbf{k}\ .$$

Further, it is useful to keep in mind the following formulae

$$[X, X^\dagger] = \int h^2(\mathbf{k}^2) B(\mathbf{k}) B^\dagger(\mathbf{k}) \mathrm{d}\mathbf{k} \tag{C.6}$$

and

$$[X, [X, X^\dagger]] = 0\ , \tag{C.7}$$

whence due to Eq. (B.8) the transformation of interest can be written as

$$W = \exp\left[g(X^\dagger - X)\right] = \exp\left(gX^\dagger\right)\ \exp\left(-gX\right)\ \exp\left(-\frac{g^2}{2}[X, X^\dagger]\right)\ . \tag{C.8}$$

Now, by using the relation (cf. Eq. (B.3))

$$a(\mathbf{k}) = W a_{\mathrm{c}}(\mathbf{k}) W^\dagger = a_{\mathrm{c}}(\mathbf{k}) - g h(\mathbf{k}^2) {B^\dagger}_{\mathrm{c}}(\mathbf{k})\ , \tag{C.9}$$

where the boson-type operator

$$B_{\mathrm{c}}(\mathbf{k}) = \int {b^\dagger}_{\mathrm{c}}(\mathbf{p} + \mathbf{k})\ b_{\mathrm{c}}(\mathbf{p}) \mathrm{d}\mathbf{p} \tag{C.10}$$

commutes with $W \equiv W(a, b) = W(a_{\mathrm{c}}, b_{\mathrm{c}})$ (see Eq. (C.4)), one can show that

$$H \equiv H(a, b) = K(a_{\mathrm{c}}, b_{\mathrm{c}}) = W H(a_{\mathrm{c}}, b_{\mathrm{c}}) W^\dagger =$$

$$= m_0 B_{\mathrm{c}}(0) + \int \omega_{\mathbf{k}} {a^\dagger}_{\mathrm{c}}(\mathbf{k}) a_{\mathrm{c}}(\mathbf{k}) \mathrm{d}\mathbf{k} - g^2 \int \omega_{\mathbf{k}} h^2(\mathbf{k}^2) {B^\dagger}_{\mathrm{c}}(\mathbf{k}) B_{\mathrm{c}}(\mathbf{k}) \mathrm{d}\mathbf{k}\ . \tag{C.11}$$

Reshuffling the fermion operators in normal order in the r.h.s. of Eq. (C.11) we obtain

$$\int \omega_{\mathbf{k}} h^2(\mathbf{k}^2) {B^\dagger}_{\mathrm{c}}(\mathbf{k}) B_{\mathrm{c}}(\mathbf{k}) \mathrm{d}\mathbf{k} = B_{\mathrm{c}}(0) \int \omega_{\mathbf{k}} h^2(\mathbf{k}^2) \mathrm{d}\mathbf{k} -$$

$$- \int \omega_{\mathbf{k}} h^2(\mathbf{k}^2) \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{p} \int \mathrm{d}\mathbf{p}'\ {b^\dagger}_{\mathrm{c}}(\mathbf{p} + \mathbf{k})\ {b^\dagger}_{\mathrm{c}}(\mathbf{p}')\ b_{\mathrm{c}}(\mathbf{p})\ b_{\mathrm{c}}(\mathbf{p}' + \mathbf{k})\ .$$

The first term in the r.h.s. of this equation has the same structure as $m_0 B_{\mathrm{c}}(0)$ in Eq. (C.11) and gives the radiative correction (renormalization) to the bare fermion mass m_0,

$$m = m_0 - g^2 \int \omega_{\mathbf{k}} h^2(\mathbf{k}^2) \mathrm{d}\mathbf{k}\ . \tag{C.12}$$

So,

$$H = K(a_{\rm c}, b_{\rm c}) = K_{\rm F} + K_{\rm I} \ , \tag{C.13}$$

$$K_{\rm F} = mB_{\rm c}(0) + \int \omega_{\mathbf{k}} a^{\dagger}{}_{\rm c}(\mathbf{k}) a_{\rm c}(\mathbf{k}) {\rm d}\mathbf{k} \equiv K_{\rm ferm} + K_{\rm boson} \ , \tag{C.14}$$

$$K_{\rm I} = \int {\rm d}\mathbf{x} \int {\rm d}\mathbf{x}' \psi^{\dagger}{}_{\rm c}(\mathbf{x}) \ \psi^{\dagger}{}_{\rm c}(\mathbf{x}') \ V_{\rm ff}(\mid \mathbf{x} - \mathbf{x}' \mid) \ \psi_{\rm c}(\mathbf{x}) \ \psi_{\rm c}(\mathbf{x}') \ , \tag{C.15}$$

$$V_{\rm ff}(\mid \mathbf{x} \mid) = -g^2 \int \omega_{\mathbf{k}} h^2(\mathbf{k}^2) {\rm e}^{i\mathbf{k}\mathbf{x}} {\rm d}\mathbf{k} \ , \tag{C.16}$$

where in agreement with the secondary quantization prescriptions we have introduced the $\psi_{\rm c}$-field for clothed fermions in the Schrödinger picture assuming

$$\psi_{\rm c}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^3}} \int b_c(\mathbf{p}) {\rm e}^{i\mathbf{p}\mathbf{x}} {\rm d}\mathbf{p}, \quad \{\psi_{\rm c}(\mathbf{x}), \psi^{\dagger}{}_{\rm c}(\mathbf{x}')\} = \delta(\mathbf{x} - \mathbf{x}').$$

Therefore, $V_{ff}(\mid \mathbf{x} \mid)$ can be considered as a two-fermion interaction potential.

One should point out that the new interaction Hamiltonian K_I expressed through clothed operators no longer contains any self-interaction and leads merely to an interaction between pairs of fermions.

The K_F eigenvectors

$$\Omega, \ a^{\dagger}{}_c(\mathbf{k}_1)\Omega, \ b^{\dagger}{}_c(\mathbf{p}_1)\Omega, \ a^{\dagger}{}_c(\mathbf{k}_1)a^{\dagger}{}_c(\mathbf{k}_2)\Omega, \ b^{\dagger}{}_c(\mathbf{p}_1)b^{\dagger}{}_c(\mathbf{p}_2)\Omega, \ a^{\dagger}{}_c(\mathbf{k}_1)b^{\dagger}{}_c(\mathbf{p}_2)\Omega, \ldots \tag{C.17}$$

with the running element

$$\Omega_c(\mathbf{k}_1 \ldots \mathbf{k}_r; \mathbf{p}_1 \ldots \mathbf{p}_s) = a^{\dagger}{}_c(\mathbf{k}_1) \ldots a^{\dagger}{}_c(\mathbf{k}_r) b^{\dagger}{}_c(\mathbf{p}_1) \ldots b^{\dagger}{}_c(\mathbf{p}_s)\Omega \ (r, s = 1, 2, \ldots) \tag{C.18}$$

form a basis (see App. B).

Another basis can be composed of bare vacuum Ω_0 and vectors

$$\Omega_0(\mathbf{k}_1 \ldots \mathbf{k}_r; \mathbf{p}_1 \ldots \mathbf{p}_s) = a^{\dagger}(\mathbf{k}_1) \ldots a^{\dagger}(\mathbf{k}_r) b^{\dagger}(\mathbf{p}_1) \ldots b^{\dagger}(\mathbf{p}_s)\Omega_0 \ (r, s = 1, 2, \ldots), \tag{C.19}$$

i.e., of the H_0 eigenstates.

Note also that in the given model, due to the absence of $f\bar{f}$-processes, the vacuum Ω for the coupled fields coincides (to an accuracy of a phase factor) with the vacuum Ω_0 for the free fields. In the context, with the help of Eq. (C.9) we find,

$$a^{\dagger}{}_c(\mathbf{k})\Omega = a^{\dagger}{}_c(\mathbf{k})\Omega_0 = a^{\dagger}(\mathbf{k})\Omega_0 \ , \tag{C.20}$$

i.e., the bare and clothed one-boson states are the same.

As to the clothed one-fermion states $b^{\dagger}{}_c(\mathbf{p})\Omega$, they are complex superpositions of vectors (C.19) with one fermion and a complicated boson configurations (a

boson cloud surrounding the fermion). This follows directly from the explicit expression for

$$b^{\dagger}{}_c(\mathbf{p}) = W^{\dagger} b^{\dagger}(\mathbf{p}) W = \int \mathcal{F}(\mathbf{p}-\mathbf{p}') b^{\dagger}(\mathbf{p}') \mathrm{d}\mathbf{p}' \ , \tag{C.21}$$

$$\mathcal{F}(\mathbf{q}) = \frac{1}{(2\pi)^3} \int \mathrm{e}^{-\imath \mathbf{q}\mathbf{x}} \exp\left\{-g \int \exp\left[-\imath \mathbf{k}\mathbf{x}\right] \left(a^{\dagger}(\mathbf{k}) - a^{\dagger}(-\mathbf{k})\right) h(\mathbf{k}^2) \mathrm{d}\mathbf{k}\right\} \mathrm{d}\mathbf{x} \ .$$

The factor $\mathcal{F}(\mathbf{q})$ characterizes a boson distribution in the cloud. In the free case (with $g = 0$) $\mathcal{F}(\mathbf{q}) = \delta(\mathbf{q})$.

C2. It follows from Eqs. (C.13)–(C.15) that clothed mesons do not interact with nucleons, they are free. In particular, no-meson states cannot pass in states with one, two, etc., mesons. This means that $K(a_c, b_c)$ possesses the property of the operator $H' = U^{\dagger} H U$ obtained from H via Okubo's transformation considered in Subsec. 6.5, viz., $H'_{12} = H'_{21} = 0$. In other words, H' has vanishing matrix elements between the no-meson states and one-, two-, ... meson states. This property of $K(a_c, b_c)$ is specific for the given simple model. In general, the clothing UT permits K to contain such nondiagonal terms that are responsible for the process $NN \to NN\pi$.

Since the model clothing UT W fulfills Okubo's requirement it is interesting to compare the block structure of $W^{\dagger}$ with that described in Subsecs. 6.2 and 6.3 for Okubo's UT. To do it in a compact form one may calculate separate blocks of $W^{\dagger}$ decomposition of the type determined by Eq. (6.14). In the context, recall that Okubo's UT can be represented by a function $U(a, b)$ of bare creation (destruction) operators. $W^{\dagger}$ can also be given by a function $W^{\dagger}(a, b)$ of the bare operators since $W(a_c, b_c) = W(a, b)$ (cf. Eq. (2.13)). Therefore, for Okubo-like decomposition of $W^{\dagger}$ one may use the projector η_1 onto bare no-meson states.

In order to carry out the comparison with a solution of the corresponding decoupling equation (6.22) (see C.3 below) we shall find the operator $\mathcal{A}^{21}$ defined by

$$W^{\dagger}{}^{21} = \mathcal{A}^{21} W^{\dagger}{}^{11} . \tag{C.22}$$

It determines the basic element $\mathcal{A}^{21}$ of Okubo's UT (see, for instance, Eqs. (6.8) and (6.20)).

We have

$$W^{\dagger}{}^{11} = \eta_1 W^{\dagger} \eta_1 = \eta_1 \exp\left(-g X^{\dagger}\right) \exp\left(-\frac{g^2}{2}\left[X, X^{\dagger}\right]\right) \exp\left(g X\right) \eta_1 =$$

$$= \exp\left(-\frac{g^2}{2}\left[X, X^{\dagger}\right]\right) \eta_1 \ , \tag{C.23}$$

$$W^{\dagger}{}^{21} = \eta_2 W^{\dagger} \eta_1 = \ \eta_2 \exp\left(-g X^{\dagger}\right) \exp\left(-\frac{g^2}{2}\left[X, X^{\dagger}\right]\right) \eta_1 =$$

$$= \eta_2 \exp\left(-gX^\dagger\right) \eta_1 {W^\dagger}^{11}. \tag{C.24}$$

While deriving these formulae we have used the property

$$a_c(\mathbf{k})\eta_1 = \eta_1 {a^\dagger}_c(\mathbf{k}) = 0, \quad \forall \mathbf{k} \tag{C.25}$$

and its consequence

$$\exp(g\,X)\eta_1 = \eta_1 = \eta_1 \exp(-g\,X^\dagger). \tag{C.26}$$

We also have taken into account the relation

$$\left[X, X^\dagger\right] \eta_1 = \eta_1 \left[X, X^\dagger\right] \ . \tag{C.27}$$

It follows from (C.23) and (C.24) that

$$\mathcal{A}^{21} = \eta_2 \exp\left(-gX^\dagger\right) \eta_1 = \left(\exp\left(-gX^\dagger\right) - 1\right) \eta_1. \tag{C.28}$$

Note that we have managed to find the operator $\mathcal{A}^{21}$ in an explicit form without solving Eq. (C.22).

Unitarity relations for the Okubo-type blocks $U^{ij} \equiv (W^\dagger)^{ij} (i, j = 1, 2)$ with $U^{21} = \mathcal{A}^{21} U^{11}$,

$$U^{11\dagger}(1 + \mathcal{A}^{21\dagger}\mathcal{A}^{21})U^{11} = \ U^{11\dagger}\left(1 + \mathcal{A}^{21\dagger}\right)\left(1 + \mathcal{A}^{21}\right) U^{11} = \eta_1 \ , \tag{C.29}$$

$$U^{12} = -\mathcal{A}^{21\dagger} U^{22} \ , \tag{C.30}$$

and

$$U^{22\dagger}(1 + \mathcal{A}^{21}\mathcal{A}^{21\dagger})U^{22} = \eta_2 \tag{C.31}$$

can directly be verified. For instance, we get step by step,

$$-\mathcal{A}^{21\dagger} U^{22} = \eta_1 \left(1 - \exp\left(-gX\right)\right) \exp\left[g(X - X^\dagger)\right] \eta_2 =$$

$$= \eta_1 \exp\left[g(X - X^\dagger)\right] \eta_2 - \eta_1 \exp\left(-gX\right) \exp\left[g(X - X^\dagger)\right] \eta_2 = {W^\dagger}^{12} \equiv U^{12} \ , \tag{C.32}$$

since in accordance with Eq. (C.27)

$$\eta_1 \exp\left(-gX\right) \exp\left[g(X - X^\dagger)\right] \eta_2 = \eta_1 \exp\left(-gX^\dagger\right) \exp\left(\frac{g^2}{2}\left[X, X^\dagger\right]\right) \eta_2 =$$

$$= \eta_1 \exp\left(\frac{g^2}{2}\left[X, X^\dagger\right]\right) \eta_2 = 0 \ ,$$

and

$$U^{22\dagger}(1 + \mathcal{A}^{21}\mathcal{A}^{21\dagger})U^{22} = \ U^{22\dagger}U^{22} + U^{12\dagger}U^{12} =$$

$$= \eta_2 \exp\left[g(X^\dagger - X)\right] \eta_2 \exp\left[g(X - X^\dagger)\right] \eta_2 +$$

$$+ \eta_2 \exp\left[g(X^\dagger - X)\right] \eta_1 \exp\left[g(X - X^\dagger)\right] \eta_2 = \eta_2 W W^\dagger \eta_2 = \eta_2 \qquad \text{(C.33)}$$

Q.E.D.

C3. Let us try to solve Okubo equation (6.22) for this model Hamiltonian. It is of great interest to attain some experience in handling with similar nonlinear equations. We prefer to start with the equation equivalent to Eq. (6.22) :

$$\eta_2\{[H_0, J_1] + V J_1 - J_1 V J_1\}\eta_1 = 0 \qquad \text{(C.34)}$$

for the operator $J_1 = (1 + \mathcal{A}^{21})\eta_1$ (cf. Eq. (29) in [7]).

One should point out the properties

$$\eta_1 J_1 = \eta_1, \qquad \text{(C.35a)}$$

$$J_1(1 - \eta_1) = 0 \qquad \text{(C.35b)}$$

and the condition

$$[H_0, \eta_1] = 0. \qquad \text{(C.36)}$$

Further, introducing the interaction picture operators O,

$$O(t) = \exp(\imath H_0 t) O \exp(-\imath H_0 t) \ ,$$

and noticing that $\eta_1(t) = \eta_1$ one can get the Riccati–type differential equation for $J_1(t)$ (cf. Eq. (31) in [7]),

$$\imath\eta_2 \frac{d}{dt} J_1(t)\eta_1 = \eta_2\{V(t)J_1(t) - J_1(t)V(t)J_1(t)\}\eta_1. \qquad \text{(C.37)}$$

In the scalar model we have

$$V(t) = g \int \omega_{\mathbf{k}} \left[B^\dagger(\mathbf{k})\, a^\dagger(\mathbf{k}, t) + \text{H.c.}\right] h(\mathbf{k}^2) \mathrm{d}\mathbf{k} \qquad \text{(C.38)}$$

with

$$a(\mathbf{k}, t) = a(\mathbf{k}) \exp(-\imath\omega_{\mathbf{k}} t) \ ,$$

so that

$$[a(\mathbf{k}, t), a^\dagger(\mathbf{k}', t')] = \delta(\mathbf{k} - \mathbf{k}') \exp[-\imath\omega_{\mathbf{k}}(t - t')]. \qquad \text{(C.39)}$$

Trying to solve Eq. (C.37), one should note the relation

$$V(t) = -\imath g \frac{d}{dt}\left[X^\dagger(t) - X(t)\right] \ , \qquad \text{(C.40)}$$

where

$$X(t) = \int h(\mathbf{k}^2) B(\mathbf{k}) a(\mathbf{k}, t) \mathrm{d}\mathbf{k} \ .$$

Now, applying the Lagrange method well known in the theory of ordinary differential equations (see, e.g., [58], §1.6) let us search a solution of Eq. (C.37) in the form

$$J_1(t) = \exp\left[-g\,X^\dagger(t)\right] G(t)\ , \tag{C.41}$$

where in accordance with Eqs. (C.35)–(C.36) we have

$$\eta_1 G(t) = \eta_1\ , \tag{C.42a}$$

$$G(t)(1-\eta_1) = 0. \tag{C.42b}$$

Differentiating (C.41) and taking into account that

$$\left[\frac{d}{dt}X^\dagger(t), X^\dagger(t)\right] = 0\ ,$$

we find

$$\frac{d}{dt}J_1(t) = -g\left(\frac{d}{dt}X^\dagger(t)\right)\exp\left[-g\,X^\dagger(t)\right] G(t) + \exp\left[-g\,X^\dagger(t)\right]\frac{d}{dt}G(t)\ .$$

Substitution of this expression into Eq. (C.37) enables us to remove in the r.h.s. of this equation not the linear term $V(t)J_1(t)$ as a whole but its part $-\imath g\frac{d}{dt}X^\dagger(t)J_1(t)$ (see Eq. (C.40)). So, we get the following equation for the operator function $G(t)$,

$$\eta_2 \exp\left[-g\,X^\dagger(t)\right]\frac{d}{dt}G(t)\eta_1 =$$

$$= g\eta_2 \exp\left[-g\,X^\dagger(t)\right]\{\frac{d}{dt}X(t) + g\left[X^\dagger(t), \frac{d}{dt}X(t)\right]\}G(t)\eta_1 -$$

$$-g\eta_2 \exp\left[-g\,X^\dagger(t)\right] G(t)\eta_1\left\{\frac{d}{dt}X(t) + g\left[X^\dagger(t), \frac{d}{dt}X(t)\right]\right\}G(t)\eta_1\ . \tag{C.43}$$

At the point we have used the properties $\eta_1 \exp\left[-g\,X^\dagger(t)\right] = \eta_1$ and $\eta_1\frac{d}{dt}X^\dagger(t) = 0$ and the relation

$$\exp\left[g\,X^\dagger(t)\right]\frac{d}{dt}X(t)\exp\left[-g\,X^\dagger(t)\right] = \frac{d}{dt}X(t) + g\left[X^\dagger(t), \frac{d}{dt}X(t)\right]\ ,$$

that follows from Eqs. (2.15) and (C.4).

Further, we have

$$\left[X^\dagger(t), \frac{d}{dt}X(t)\right] = \imath\int \omega_{\mathbf{k}} h^2(\mathbf{k}^2) B^\dagger(\mathbf{k})B(\mathbf{k})\mathrm{d}\mathbf{k}\ \equiv C.$$

In other words, this time independent commutator acts merely on the fermionic degrees of freedom. Note that C commutes with the projector η_1.

Equation (C.43) can be satisfied if we put

$$\frac{d}{dt}G(t) = g\{\frac{d}{dt}X(t) + gC\}G(t) - gG(t)\{\frac{d}{dt}X(t) + gC\}G(t) \ . \qquad (C.44)$$

It may be shown that one of its possible solutions is

$$G_1(t) = \eta_1. \qquad (C.45)$$

It is evident that $G_1(t)$ meets necessary requirements (C.42).

The corresponding operator $J_1(t) = \exp\left[-g\,X^\dagger(t)\right]\eta_1$ yields

$$\mathcal{A}^{21}(t) = \{\exp\left[-g\,X^\dagger(t)\right] - 1\}\eta_1 \ , \qquad (C.46)$$

that is equivalent at $t = 0$ to the result (C.28).

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