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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

NOTOPH GAUGE THEORY: SUPERFIELD FORMALISM

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We derive absolutely anticommuting Becchi–Rouet–Stora–Tyutin (BRST) and anti-BRST symmetry transformations for the 4D free Abelian 2-form gauge theory by exploiting the superfield approach to BRST formalism. The antisymmetric tensor gauge field of the above theory was christened as the «notoph» (i.e., the opposite of «photon») gauge field by Ogievetsky and Polubarinov way back in 1966–67. We briefly outline the problems involved in obtaining the absolute anticommutativity of the (anti-)BRST transformations and their resolution within the framework of geometrical superfield approach to BRST formalism. One of the highlights of our results is the emergence of a Curci–Ferrari type of restriction in the context of 4D Abelian 2-form (notoph) gauge theory which renders the nilpotent (anti-)BRST symmetries of the theory to be absolutely anticommutative in nature.

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INTRODUCTION

The Becchi–Rouet–Stora–Tyutin (BRST) and anti-BRST symmetry transformations emerge when the «classical» local gauge symmetry transformations of any arbitrary p-form ($p = 1, 2, 3, \ldots$) gauge theory are elevated to the «quantum» level. The above (anti-)BRST symmetry transformations are found to be nilpotent of order two and they anticommute with each other. These properties are very sacrosanct as they encode (i) the fermionic nature of these symmetries and (ii) the linear independence of these transformations (see, e.g., [1]). These statements are true for the BRST approach to any arbitrary p-form gauge theories in any arbitrary dimension of space-time.

In recent years, the Abelian 2-form (i.e., $B^{(2)} = (1/2!)(dx^\mu \wedge dx^\nu)B_{\mu\nu}$) gauge theory with antisymmetric ($B_{\mu\nu} = -B_{\nu\mu}$) tensor potential[2] $B_{\mu\nu}$ has become quite popular because of its relevance in the context of (super)string theories [3, 4] and the noncommutativity associated with them due to the presence of $B_{\mu\nu}$ in the background [5]. It has been shown, furthermore, that the Abelian 2-form (notoph) gauge theory provides a tractable field-theoretic model for the Hodge theory [6–8] as well as quasi-topological field theory [9]. This theory has been discussed within the framework of the BRST formalism, too (see, e.g., [10–12]). The known

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[2]This potential was christened as the notoph gauge field by Ogievetsky and Polubarinov, who were the first to discuss this gauge theory at BLTP, JINR, Dubna [2].

nilpotent (anti-)BRST transformations, however, have been found to be anticommuting only up to the $U(1)$ vector gauge transformations (see, e.g., [6, 7] for details).

One of the key problems in the context of 4D notoph gauge theory has been to obtain a set of (anti-)BRST symmetry transformations that are consistent with the basic tenets of BRST formalism. The central theme of our presentation is to obtain *absolutely* anticommuting off-shell nilpotent (anti-)BRST symmetry transformations for the notoph gauge theory by exploiting the geometrical superfield approach to BRST formalism proposed by Bonora and Tonin [13, 14]. We demonstrate that a Curci–Ferrari (CF) type restriction emerges from the superfield formalism which enables us to derive (i) the absolute anticommutativity of the (anti-)BRST symmetry transformations and (ii) the coupled Lagrangian densities of the theory that respect these (anti-)BRST symmetry transformations. The idea of the horizontality condition (HC) is at the heart of these derivations.

The layout of our presentation is as follows. First, we recapitulate the bare essentials of the nilpotent (anti-)BRST symmetry transformations [6, 7] that are anticommuting only up to a vector $U(1)$ gauge transformation. We describe, after this, the key issues associated with the HC within the framework of the superfield formalism. Next we derive the CF-type restriction by exploiting the celebrated HC. The former turns out to be (anti-)BRST invariant quantity, and it leads to the derivation of the coupled Lagrangian densities for the notoph gauge theory. These Lagrangian densities, in turn, respect the off-shell nilpotent and absolutely anicommuting (anti-)BRST symmetry transformations. Finally, we provide geometrical interpretations for the nilpotent and anticommuting symmetries (and corresponding generators) within the framework of superfield approach to BRST formalism.

1. PRELIMINARIES: OLD LAGRANGIAN FORMULATION AND OFF-SHELL NILPOTENT SYMMETRIES

We begin with the generalized version of the Kalb–Ramond Lagrangian density ($\mathcal{L}^{(0)} = (1/12)H^{\mu\nu\kappa}H_{\mu\nu\kappa}$) for the 4D[1] notoph gauge theory that respects the off-shell nilpotent (anti-)BRST transformations [6, 7]. This Lagrangian density, in its full blaze of glory, is as follows (see, e.g., [6, 7] for details):

$$\mathcal{L}_B^{(0)} = \frac{1}{12}H^{\mu\nu\kappa}H_{\mu\nu\kappa} + B^\mu(\partial^\nu B_{\nu\mu} - \partial_\mu\phi) - \frac{1}{2}B\cdot B - \partial_\mu\bar{\beta}\partial^\mu\beta + \\ + (\partial_\mu\bar{C}_\nu - \partial_\nu\bar{C}_\mu)(\partial^\mu C^\nu) + \rho(\partial\cdot C + \lambda) + (\partial\cdot\bar{C} + \rho)\lambda, \quad (1)$$

where $B_\mu = \partial^\nu B_{\nu\mu} - \partial_\mu\phi$ is the Lorentz vector auxiliary field that has been invoked to linearize the gauge-fixing term, the massless ($\Box\phi = 0$) scalar field ϕ has been introduced for the stage-one reducibility in the theory and the totally antisymmetric curvature tensor $H_{\mu\nu\kappa} = \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu}$ is constructed with the help of the notoph gauge field $B_{\mu\nu}$.

The fermionic (i.e., $C_\mu^2 = \bar{C}_\mu^2 = 0, C_\mu\bar{C}_\nu + \bar{C}_\nu C_\mu = 0$, etc.) Lorentz vector (anti-)ghost fields $(\bar{C}_\mu)C_\mu$ (carrying ghost number $(-1)1$) have been introduced to compensate for the

[1]We choose, for the whole body of our present text, the 4D flat metric $\eta_{\mu\nu}$ with signature $(+1, -1, -1, -1)$, where the Greek indices $\mu, \nu, \eta \ldots = 0, 1, 2, 3$. The convention $(\delta B_{\mu\nu}/\delta B_{\eta\kappa}) = (1/2!)(\delta_{\mu\eta}\delta_{\nu\kappa} - \delta_{\mu\kappa}\delta_{\nu\eta})$ has been adopted in our full text [6, 7].

above gauge-fixing term, and they play important roles in the existence of the (anti-)BRST symmetry transformations for the notoph gauge potential. The bosonic (anti-)ghost fields $(\bar{\beta})\beta$ (carrying ghost numbers $(-2)2$) are needed for the requirement of ghost-for-ghost in the theory. The auxiliary ghost fields $\rho = -(1/2)(\partial \cdot \bar{C})$ and $\lambda = -(1/2)(\partial \cdot C)$ (with ghost numbers (–1)1) are also present in the theory.

The following off-shell nilpotent ($\tilde{s}^2_{(a)b} = 0$) (anti-)BRST symmetry transformations $\tilde{s}_{(a)b}$ for the fields of the Lagrangian density (1):

$$\begin{array}{lll} \tilde{s}_b B_{\mu\nu} = -(\partial_\mu C_\nu - \partial_\nu C_\mu), & \tilde{s}_b C_\mu = -\partial_\mu \beta, & \tilde{s}_b \bar{C}_\mu = -B_\mu, \\ \tilde{s}_b \phi = \lambda, & \tilde{s}_b \bar{\beta} = -\rho, & \tilde{s}_b [\rho, \lambda, \beta, B_\mu, H_{\mu\nu\kappa}] = 0, \\ \tilde{s}_{ab} B_{\mu\nu} = -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), & \tilde{s}_{ab} \bar{C}_\mu = +\partial_\mu \bar{\beta}, & \tilde{s}_{ab} C_\mu = +B_\mu, \\ \tilde{s}_{ab} \phi = \rho, & \tilde{s}_{ab} \beta = -\lambda, & \tilde{s}_{ab} [\rho, \lambda, \bar{\beta}, B_\mu, H_{\mu\nu\kappa}] = 0, \end{array} \tag{2}$$

leave the Lagrangian density (1) quasi-invariant because it changes to the total space-time derivatives as given below:

$$\begin{aligned} \tilde{s}_b \mathcal{L}_B^{(0)} &= -\partial_\mu \big[B^\mu \lambda + (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\mu - \rho \partial^\mu \beta \big], \\ \tilde{s}_{ab} \mathcal{L}_B^{(0)} &= -\partial_\mu \big[B^\mu \rho + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\mu - \lambda \partial^\mu \bar{\beta} \big]. \end{aligned} \tag{3}$$

Thus, the action corresponding to the Lagrangian density (1) remains invariant under the off-shell nilpotent (anti-)BRST transformations (2).

It can be checked that $(\tilde{s}_b \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_b) C_\mu = \partial_\mu \lambda$ and $(\tilde{s}_b \tilde{s}_{ab} + \tilde{s}_{ab} \tilde{s}_b) \bar{C}_\mu = -\partial_\mu \rho$. The above anticommutator for the rest of the fields, however, turns out to be absolutely zero. Thus, we note that the (anti-)BRST transformations are anticommuting only up to the $U(1)$ vector gauge transformations. They are *not* absolutely anticommuting for fields C_μ and $\bar{C}_\mu$. In other words, the off-shell nilpotent (anti-)BRST symmetry transformations (2) are *not* consistent with the basic tenets of BRST formalism.

2. HORIZONTALITY CONDITION: A SYNOPSIS

The off-shell nilpotency and absolute anticommutativity properties of the (anti-)BRST symmetry transformations are the natural consequences of the application of the superfield approach to BRST formalism [13–16]. Thus, we take recourse to this formalism to resolve the problem that has been stated earlier. In fact, we derive the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for the notoph gauge theory by exploiting the celebrated horizontality condition within the framework of the geometrical superfield approach [13–16].

The notoph gauge theory is endowed with the first-class constraints [17] in the language of Dirac's prescription for the classification scheme. As a consequence, the theory respects a local gauge symmetry transformation that is generated by these constraints. The above classical local symmetry transformation is traded with the (anti-)BRST symmetry transformations at the quantum level. The latter can be derived by exploiting the superfield formalism [18]. One of the key ingredients in the superfield formulation is to consider the 4D ordinary gauge theory on a $(4,2)$-dimensional supermanifold where one has the following $N = 2$ generalizations [18]:

$$
\begin{aligned}
x^\mu \to Z^M = (x^\mu, \theta, \bar{\theta}), \quad d = dx^\mu \partial_\mu \to \tilde{d} = dx^\mu \partial_\mu + d\theta \partial_\theta + d\bar{\theta} \partial_{\bar{\theta}}, \\
B^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu) B_{\mu\nu} \to \tilde{B}^{(2)} = \frac{1}{2!}(dZ^M \wedge dZ^N) B_{MN}.
\end{aligned}
\tag{4}
$$

In the above, $Z^M = (x^\mu, \theta, \bar{\theta})$ is the $N = 2$ superspace variable, θ and $\bar{\theta}$ are the Grassmannian variables (with $\theta^2 = \bar{\theta}^2 = 0$, $\theta\bar{\theta} + \bar{\theta}\theta = 0$), $\tilde{d} = dZ^M \partial_M$ is the super exterior derivative (with $\partial_M = (\partial_\mu, \partial_\theta, \partial_{\bar{\theta}})$), and $\tilde{B}^{(2)}$ is the super 2-form (notoph) gauge field connection with a few multiplet superfields.

The explicit form of the above super 2-form connection field is as follows:

$$
\begin{aligned}
\tilde{B}^{(2)} = \frac{1}{2!}(dx^\mu \wedge dx^\nu) \tilde{B}_{\mu\nu}(x, \theta, \bar{\theta}) + (dx^\mu \wedge d\theta) \tilde{\bar{\mathcal{F}}}_\mu(x, \theta, \bar{\theta}) + (dx^\mu \wedge d\bar{\theta}) \tilde{\mathcal{F}}_\mu(x, \theta, \bar{\theta}) + \\
+ (d\theta \wedge d\theta) \tilde{\bar{\beta}}(x, \theta, \bar{\theta}) + (d\bar{\theta} \wedge d\bar{\theta})\, \tilde{\beta}(x, \theta, \bar{\theta}) + (d\theta \wedge d\bar{\theta}) \tilde{\Phi}(x, \theta, \bar{\theta}).
\end{aligned}
\tag{5}
$$

In the above, the (4, 2)-dimensional multiplet superfields (see, e.g., [18])

$$
\tilde{B}_{\mu\nu}(x, \theta, \bar{\theta}), \quad \tilde{\bar{\mathcal{F}}}_\mu(x, \theta, \bar{\theta}), \quad \tilde{\mathcal{F}}_\mu(x, \theta, \bar{\theta}), \quad \tilde{\bar{\beta}}(x, \theta, \bar{\theta}), \quad \tilde{\beta}(x, \theta, \bar{\theta}), \quad \tilde{\Phi}(x, \theta, \bar{\theta}) \tag{6}
$$

are the generalizations of the basic local fields $B_{\mu\nu}, \bar{C}_\mu, C_\mu, \bar{\beta}, \beta, \phi$ of the nilpotent (anti-)BRST invariant Lagrangian density (1) of the 4D notoph gauge theory. This can be explicitly seen by the following super expansion of these superfields along the Grassmannian directions of the supermanifold:

$$
\begin{aligned}
\tilde{B}_{\mu\nu}(x, \theta, \bar{\theta}) &= B_{\mu\nu}(x) + \theta \bar{R}_{\mu\nu}(x) + \bar{\theta} R_{\mu\nu}(x) + i\theta\bar{\theta} S_{\mu\nu}(x), \\
\tilde{\beta}(x, \theta, \bar{\theta}) &= \beta(x) + \theta \bar{f}_1(x) + \bar{\theta} f_1(x) + i\theta\bar{\theta} b_1(x), \\
\tilde{\bar{\beta}}(x, \theta, \bar{\theta}) &= \bar{\beta}(x) + \theta \bar{f}_2(x) + \bar{\theta} f_2(x) + i\theta\bar{\theta} b_2(x), \\
\tilde{\Phi}(x, \theta, \bar{\theta}) &= \phi(x) + \theta \bar{f}_3(x) + \bar{\theta} f_3(x) + i\theta\bar{\theta} b_3(x), \\
\tilde{\mathcal{F}}_\mu(x, \theta, \bar{\theta}) &= C_\mu(x) + \theta \bar{B}^{(1)}_\mu(x) + \bar{\theta} B^{(1)}_\mu(x) + i\theta\bar{\theta} f^{(1)}_\mu(x), \\
\tilde{\bar{\mathcal{F}}}_\mu(x, \theta, \bar{\theta}) &= \bar{C}_\mu(x) + \theta \bar{B}^{(2)}_\mu(x) + \bar{\theta} B^{(2)}_\mu(x) + i\theta\bar{\theta} \bar{f}^{(2)}_\mu(x).
\end{aligned}
\tag{7}
$$

In the limit $(\theta, \bar{\theta}) \to 0$, we retrieve our basic local fields of the original 4D notoph gauge theory. Furthermore, the above expansion is in terms of the basic fields (6) and rest of the fields in the expansion are secondary fields.

To obtain the explicit form of the secondary fields in terms of the basic fields, one has to invoke the celebrated HC (i.e., $\tilde{d}\tilde{B}^{(2)} = dB^{(2)}$) which is the requirement that the curvature 3-form $H^{(3)} = dB^{(2)}$ remains unaffected by the presence of the supersymmetry in the theory.

In other words, all the Grassmannian components of the following super 3-form:

$$\begin{aligned}
\tilde{d}\tilde{B}^{(2)} = \frac{1}{2!}(dx^\kappa \wedge dx^\mu \wedge dx^\nu)(\partial_\kappa \tilde{\mathcal{B}}_{\mu\nu}) + (d\theta \wedge d\theta \wedge d\theta)(\partial_\theta \tilde{\bar{\beta}})+ \\
+ (d\theta \wedge d\bar{\theta} \wedge d\bar{\theta})\big[\partial_{\bar{\theta}}\tilde{\Phi} + \partial_\theta \tilde{\beta}\big] + (d\bar{\theta} \wedge d\theta \wedge d\theta)\big[\partial_\theta \tilde{\Phi} + \partial_{\bar{\theta}}\tilde{\bar{\beta}}\big]+ \\
+ \frac{1}{2!}(dx^\mu \wedge dx^\nu \wedge d\theta)\big[\partial_\theta \tilde{\mathcal{B}}_{\mu\nu} + \partial_\mu \tilde{\bar{\mathcal{F}}}_\nu - \partial_\nu \tilde{\bar{\mathcal{F}}}_\mu\big]+ \\
+ (dx^\mu \wedge d\theta \wedge d\theta)\big[\partial_\theta \tilde{\bar{\mathcal{F}}}_\mu + \partial_\mu \tilde{\bar{\beta}}\big] + (dx^\mu \wedge d\bar{\theta} \wedge d\bar{\theta})\big[\partial_{\bar{\theta}}\tilde{\mathcal{F}}_\mu + \partial_\mu \tilde{\beta}\big]+ \\
+ \frac{1}{2!}(dx^\mu \wedge dx^\nu \wedge d\bar{\theta})\big[\partial_{\bar{\theta}}\tilde{\mathcal{B}}_{\mu\nu} + \partial_\mu \tilde{\mathcal{F}}_\nu - \partial_\nu \tilde{\mathcal{F}}_\mu\big]+ \\
+ (dx^\mu \wedge d\theta \wedge d\bar{\theta})\big[\partial_\mu \tilde{\Phi} + \partial_\theta \tilde{\mathcal{F}}_\mu + \partial_{\bar{\theta}}\tilde{\bar{\mathcal{F}}}_\mu\big] + (d\bar{\theta} \wedge d\bar{\theta} \wedge d\bar{\theta})(\partial_{\bar{\theta}}\tilde{\beta}) \quad (8)
\end{aligned}$$

are to be set equal to zero. This condition has been referred to as the soul-flatness condition by Nakanashi and Ojima [19].

Physically, the soul-flatness condition (or HC) is the requirement that the gauge (i.e., (anti-)BRST) invariant quantity (i.e., curvature tensor) should remain independent of the Grassmannian coordinates that are present in the superspace variable $Z^M = (x^\mu, \theta, \bar{\theta})$. This is evident from Eq. (2) where we note that $\tilde{s}_{(a)b} H_{\mu\nu\kappa} = 0$. The celebrated HC, we emphasize once again, always leads to the symmetry transformations that are nilpotent and absolutely anticommuting because these are the properties that are associated with the Grassmannian variables that play a very important role in HC. We shall be able to see these consequences in the next section.

3. CURCI–FERRARI TYPE RESTRICTION AND SUPERFIELD EXPANSIONS: SUPERFIELD FORMALISM

As a consequence of the HC, we can set the coefficients of the 3-form differentials $(d\theta \wedge d\theta \wedge d\theta)$, $(d\bar{\theta} \wedge d\bar{\theta} \wedge d\bar{\theta})$, $(d\theta \wedge d\theta \wedge d\bar{\theta})$, $(d\theta \wedge d\bar{\theta} \wedge d\bar{\theta})$ equal to zero. These requirements lead to the following conditions on some of the secondary fields that are present in the expansions of the superfields:

$$f_1 = \bar{f}_2 = b_1 = b_2 = b_3 = 0, \qquad f_2 + \bar{f}_3 = 0, \qquad \bar{f}_1 + f_3 = 0. \quad (9)$$

In an exactly similar fashion, setting the coefficients of the differentials $(dx^\mu \wedge dx^\nu \wedge d\theta), (dx^\mu \wedge dx^\nu \wedge d\bar{\theta}), (dx^\mu \wedge d\theta \wedge d\theta), (dx^\mu \wedge d\bar{\theta} \wedge d\bar{\theta})$ equal to zero, we obtain the following conditions on some of the secondary fields [18]:

$$\begin{aligned}
B_\mu^{(1)} = -\partial_\mu \beta, \quad \bar{B}_\mu^{(2)} = -\partial_\mu \bar{\beta}, \quad f_\mu^{(1)} = i\partial_\mu \lambda, \quad \bar{f}_\mu^{(2)} = -i\partial_\mu \rho, \\
R_{\mu\nu} = -(\partial_\mu C_\nu - \partial_\nu C_\mu), \quad \bar{R}_{\mu\nu} = -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \\
S_{\mu\nu} = -i(\partial_\mu B_\nu - \partial_\nu B_\mu) \equiv -i(\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu),
\end{aligned} \quad (10)$$

where we have identified $\bar{B}_\mu^{(1)} = \bar{B}_\mu, B_\mu^{(2)} = -B_\mu$.

Finally, it is very interesting to point out that we obtain the (anti-)BRST invariant Curci–Ferrari (CF) type restriction

$$B_\mu - \bar{B}_\mu - \partial_\mu \phi = 0, \quad (11)$$

when we set the coefficient of the 3-form differential $(dx^\mu \wedge d\theta \wedge d\bar{\theta})$ equal to zero (due to the celebrated HC). It would be worthwhile to state that one encounters such kind of restriction in the case of 4D non-Abelain 1-form gauge theory [20] which enables one to obtain the absolute anticommutativity of the off-shell nilpotent (anti-)BRST symmetry transformations. The derivation of the CF restriction [20] within the framework of superfield formalism (in the context of the 4D non-Abelian 1-form gauge theory) has been performed by Bonora and Tonin (see, e.g., [13] for details).

The stage is now set for the comparison of the coefficient of the 3-form differential $(dx^\mu \wedge dx^\nu \wedge dx^\kappa)$ from the l.h.s. and r.h.s of the horizontality condition $\tilde{d}\tilde{B}^{(2)} = dB^{(2)}$ where the r.h.s. produces $(1/3!)(dx^\mu \wedge dx^\nu \wedge dx^\kappa)H_{\mu\nu\kappa}$ only. However, there are terms with Grassmannian variables on the l.h.s. Setting these terms equal to zero leads to

$$\begin{aligned} \partial_\mu R_{\nu\kappa} + \partial_\nu R_{\kappa\mu} + \partial_\kappa R_{\mu\nu} = 0, \\ \partial_\mu \bar{R}_{\nu\kappa} + \partial_\nu \bar{R}_{\kappa\mu} + \partial_\kappa \bar{R}_{\mu\nu} = 0, \\ \partial_\mu S_{\nu\kappa} + \partial_\nu S_{\kappa\mu} + \partial_\kappa S_{\mu\nu} = 0, \end{aligned} \tag{12}$$

which are automatically satisfied due to values in Eq. (10).

Let us focus on the expansion of the superfield $\tilde{\mathcal{B}}_{\mu\nu}(x, \theta, \bar{\theta})$ with the values that are given in (10). We obtain the following:

$$\tilde{\mathcal{B}}_{\mu\nu}(x, \theta, \bar{\theta}) = B_{\mu\nu}(x) - \theta(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) - \bar{\theta}(\partial_\mu C_\nu - \partial_\nu C_\mu) + \theta\bar{\theta}(\partial_\mu B_\nu - \partial_\nu B_\mu). \tag{13}$$

Having our knowledge of the local gauge symmetry and corresponding nilpotent (anti-)BRST symmetries, we know that the coefficient of θ in the above is nothing but the anti-BRST symmetry transformation and that of $\bar{\theta}$ is the BRST symmetry transformation. We can now guess that the coefficient of $\theta\bar{\theta}$ should be the anticommutator of (anti-)BRST symmetry transformations because of the anticommuting properties associated with the Grassmannian variables. Finally, it can be seen that we have the following expansion:

$$\tilde{\mathcal{B}}^{(h)}_{\mu\nu}(x, \theta, \bar{\theta}) = B_{\mu\nu}(x) + \theta(s_{ab} B_{\mu\nu}(x)) + \bar{\theta}(s_b B_{\mu\nu}(x)) + \theta\bar{\theta}(s_b s_{ab} B_{\mu\nu}(x)), \tag{14}$$

where the superscript (h) denotes the expansion of the gauge superfield $\tilde{\mathcal{B}}_{\mu\nu}(x, \theta, \bar{\theta})$ after the application of HC and symbols $s_{(a)b}$ correspond to the *correct* (anti-)BRST symmetry transformations that are always nilpotent of order two and absolutely anticommuting in nature.

The substitution of all the values of the secondary fields from (10) leads to the following expansion of the rest of the superfields of (7):

$$\begin{aligned} \tilde{\beta}^{(h)}(x, \theta, \bar{\theta}) &= \beta(x) + \theta(s_{ab}\beta(x)) + \bar{\theta}(s_b\beta(x)) + \theta\bar{\theta}(s_b s_{ab}\beta(x)), \\ \tilde{\bar{\beta}}^{(h)}(x, \theta, \bar{\theta}) &= \bar{\beta}(x) + \theta(s_{ab}\bar{\beta}(x)) + \bar{\theta}(s_b\bar{\beta}(x)) + \theta\bar{\theta}(s_b s_{ab}\bar{\beta}(x)), \\ \tilde{\Phi}^{(h)}(x, \theta, \bar{\theta}) &= \phi(x) + \theta(s_{ab}\phi(x)) + \bar{\theta}(s_b\phi(x)) + \theta\bar{\theta}(s_b s_{ab}\phi(x)), \\ \tilde{\mathcal{F}}^{(h)}_\mu(x, \theta, \bar{\theta}) &= C_\mu(x) + \theta(s_{ab}C_\mu(x)) + \bar{\theta}(s_b C_\mu(x)) + \theta\bar{\theta}(s_b s_{ab}C_\mu(x)), \\ \tilde{\bar{\mathcal{F}}}^{(h)}_\mu(x, \theta, \bar{\theta}) &= \bar{C}_\mu(x) + \theta(s_{ab}\bar{C}_\mu(x)) + \bar{\theta}(s_b\bar{C}_\mu(x)) + \theta\bar{\theta}(s_b s_{ab}\bar{C}_\mu(x)). \end{aligned} \tag{15}$$

Thus, we have obtained the absolutely anticommuting (anti-)BRST symmetry transformations for the notoph gauge theory as

$$\begin{gathered} s_b B_{\mu\nu} = -(\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_b C_\mu = -\partial_\mu \beta, \quad s_b \bar{C}_\mu = -B_\mu, \\ s_b \phi = \lambda, \quad s_b \bar{\beta} = -\rho, \quad \tilde{s}_b [\rho, \lambda, \beta, B_\mu, H_{\mu\nu\kappa}] = 0, \\ s_{ab} B_{\mu\nu} = -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab} \bar{C}_\mu = -\partial_\mu \bar{\beta}, \quad s_{ab} C_\mu = +\bar{B}_\mu, \\ s_{ab} \phi = \rho, \quad s_{ab} \beta = -\lambda, \quad \tilde{s}_{ab} [\rho, \lambda, \bar{\beta}, B_\mu, H_{\mu\nu\kappa}] = 0, \end{gathered} \tag{16}$$

which are different from earlier nilpotent transformations (2).

It can be checked that $(s_b s_{ab} + s_{ab} s_b) B_{\mu\nu}(x) = 0$ is true if and only if we impose the CF-type restriction (11) that has emerged out from the application of superfield formalism to the notoph gauge theory. Furthermore, the absolute anticommutativity criterion dictates the (anti-)BRST symmetry transformations on the auxiliary fields B_μ and $\bar{B}_\mu$ as

$$s_b \bar{B}_\mu = -\partial_\mu \lambda, \quad s_{ab} B_\mu = -\partial_\mu \rho, \quad s_b B_\mu = 0, \quad s_{ab} \bar{B}_\mu = 0. \tag{17}$$

Under the off-shell nilpotent (anti-)BRST symmetry transformations (16) and (17), it can be seen that the absolute anticommutativity is satisfied for all the fields of the theory which can be generically expressed as

$$\{s_b, s_{ab}\} \Omega = 0, \quad \Omega = C_\mu, \bar{C}_\mu, \beta, \bar{\beta}, B_\mu, \bar{B}_\mu, \rho, \lambda, \phi, \tag{18}$$

where Ω is the generic local field of the 4D notoph theory.

4. COUPLED LAGRANGIAN DENSITIES: DERIVATION FROM (ANTI-)BRST APPROACH

With the (anti-)BRST symmetry transformations (listed in (16) and (17)), it can be seen that the Lagrangian density for the theory can be written in two different ways. These are as follows:

$$\begin{aligned} \mathcal{L}_B &= \frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + s_b s_{ab} \Big[2\beta\bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \Big], \\ \mathcal{L}_{\bar{B}} &= \frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} - s_{ab} s_b \Big[2\beta\bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \Big], \end{aligned} \tag{19}$$

where the first term is nothing but the kinetic term for the notoph gauge field which is automatically gauge (and, therefore, (anti-)BRST)) invariant. The explicit form of the bracketed terms are

$$\begin{aligned} s_b s_{ab} \Big[2\beta\bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \Big] &= B^\mu (\partial^\nu B_{\nu\mu}) + B \cdot \bar{B} + \\ &+ \partial_\mu \bar{\beta} \partial^\mu \beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\lambda, \end{aligned} \tag{20}$$

$$\begin{aligned} - s_{ab} s_b \Big[2\beta\bar{\beta} + \bar{C}_\mu C^\mu - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \Big] &= \bar{B}^\mu (\partial^\nu B_{\nu\mu}) + B \cdot \bar{B} + \\ &+ \partial_\mu \bar{\beta} \partial^\mu \beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\lambda. \end{aligned} \tag{21}$$

It is to be noted that the difference between the above two expressions are only in the first term. However, modulo a total space-time derivative, these terms are equivalent because of the CF-type restriction in (11). Thus, the Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ are coupled (but equivalent) Lagrangian densities for the notoph gauge theory in four dimensions of space-time.

Due to CF-type relation (11), we can have the following expressions for the term $(B \cdot \bar{B})$ that appears on the r.h.s. of Eqs. (20) and (21):

$$B \cdot \bar{B} = B \cdot B - B^\mu \partial_\mu \phi, \quad B \cdot \bar{B} = \bar{B} \cdot \bar{B} + \bar{B}^\mu \partial_\mu \phi. \tag{22}$$

As a consequence of the above equations, we have the following:

$$\begin{aligned} \mathcal{L}_B &= \frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + B^\mu (\partial^\nu B_{\nu\mu} - \partial_\mu \phi) + B \cdot B + \partial_\mu \bar{\beta} \partial^\mu \beta + \\ &\qquad + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\lambda, \\ \mathcal{L}_{\bar{B}} &= \frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + \bar{B}^\mu (\partial^\nu B_{\nu\mu} + \partial_\mu \phi) + \bar{B} \cdot \bar{B} + \partial_\mu \bar{\beta} \partial^\mu \beta + \\ &\qquad + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda)\rho + (\partial \cdot \bar{C} + \rho)\lambda, \end{aligned} \tag{23}$$

which lead to the Euler–Lagrange equations of motion

$$B_\mu = -\frac{1}{2}(\partial^\nu B_{\nu\mu} - \partial_\mu \phi), \quad \bar{B}_\mu = -\frac{1}{2}(\partial^\nu B_{\nu\mu} + \partial_\mu \phi), \tag{24}$$

which imply the CF-type condition in (11).

The coupled Lagrangian densities, which have been derived due to the techniques of the (anti-)BRST formalism and use of the CF-type restriction (11) (emerging from the superfield formalism) are found to be quasi-invariant under the (anti-)BRST symmetry transformations (17) and (16). This can be seen from the following equations:

$$\begin{aligned} s_b \mathcal{L}_B &= -\partial_\mu \big[B^\mu \lambda + (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\mu + \rho \partial^\mu \beta \big], \\ s_{ab} \mathcal{L}_{\bar{B}} &= -\partial_\mu \big[(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \bar{B}_\mu + \lambda \partial^\mu \bar{\beta} - \rho \bar{B}^\mu \big], \end{aligned} \tag{25}$$

which establish that the action remains invariant under (16) and (17).

One would be curious to know the transformation properties of the Lagrangian density $\mathcal{L}_B$ under the anti-BRST transformations s_{ab} and that of $\mathcal{L}_{\bar{B}}$ under the transformations s_b. It is very interesting to check that, under s_{ab}, the Lagrangian density $\mathcal{L}_B$ transforms to a total space-time derivative plus terms that are zero on the constrained surface defined by the field equation (11). Similar is the situation of $\mathcal{L}_{\bar{B}}$ under the transformations s_b. Thus, we conclude that the superfield formalism provides the (anti-)BRST symmetry transformations, CF-type restriction (11) and ensuing coupled Lagrangian densities for the notoph gauge theory (see, e.g., [8] and [18]).

5. GEOMETRICAL MEANING: SUPERFIELD APPROACH

We concisely pin-point here the geometrical meaning of the (anti-)BRST symmetry transformations and the mathematical properties associated with them. In fact, one can encapsulate the geometrical interpretations in the language of the following mathematical mappings:

$$\begin{gathered}
s_b \Leftrightarrow Q_b \Leftrightarrow \lim_{\theta \to 0} \frac{\partial}{\partial \bar\theta}, \quad s_{ab} \Leftrightarrow Q_{ab} \Leftrightarrow \lim_{\bar\theta \to 0} \frac{\partial}{\partial \theta}, \\
s_b^2 = 0 \Leftrightarrow Q_b^2 = 0 \Leftrightarrow \lim_{\theta \to 0} \left(\frac{\partial}{\partial \bar\theta} \right)^2 = 0, \\
s_{ab}^2 = 0 \Leftrightarrow Q_{ab}^2 = 0 \Leftrightarrow \lim_{\bar\theta \to 0} \left(\frac{\partial}{\partial \theta} \right)^2 = 0, \qquad (26) \\
s_b s_{ab} + s_{ab} s_b = 0 \Leftrightarrow Q_b Q_{ab} + Q_{ab} Q_b = 0 \Leftrightarrow \\
\Leftrightarrow \left(\lim_{\bar\theta \to 0} \frac{\partial}{\partial \theta} \right) \left(\lim_{\theta \to 0} \frac{\partial}{\partial \bar\theta} \right) + \left(\lim_{\theta \to 0} \frac{\partial}{\partial \bar\theta} \right) \left(\lim_{\bar\theta \to 0} \frac{\partial}{\partial \theta} \right) = 0.
\end{gathered}$$

The above (geometrically intuitive) mappings are possible only in the superfield approach to BRST formalism proposed in [13–16] where $Q_{(a)b}$ are the nilpotent (anti-)BRST charges corresponding to $s_{(a)b}$.

The first line in (26) implies that the off-shell nilpotent (anti-)BRST symmetry transformations $s_{(a)b}$ and their corresponding generators $Q_{(a)b}$ geometrically correspond to the translational generators along the Grassmannian directions of the (4, 2)-dimensional supermanifold. To be more specific, the BRST symmetry transformation corresponds to the translation of the particular superfield along the $\bar\theta$ direction of the supermanifold when there is no translation of the same superfield along the θ direction of the supermanifold (i.e., $\theta \to 0$). This geometrical operation on the specific superfield generates the BRST symmetry transformation for the corresponding 4D ordinary field present in the Lagrangian densities (23). A similar kind of argument can be provided for the existence of the anti-BRST symmetry transformation for a specific field in the language of the translational generator (i.e., $\lim_{\bar\theta \to 0} (\partial/\partial\theta)$) on the above $(4, 2)$-dimensional supermanifold.

CONCLUSIONS

It is evident that the superfield approach to BRST formalism [13, 14] is an essential theoretical tool that always leads to the derivation of the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations for a given 4D p-form gauge theory [18]. In addition, it provides the geometrical origin and interpretation for the properties of nilpotency and absolute anticommutativity in the language of translational generators along the Grassmannian directions of the (4, 2)-dimensional supermanifold. In our very recent work [21], we have been able to apply the superfield formalism to 4D Abelian 3-form gauge theory and we have shown the existence of the CF-type restrictions that are deeply connected with the idea of gerbes.

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