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RELATIVISTIC ASPECTS OF THE MOTION OF TWO-PARTICLE MODEL MOLECULE OSCILLATING PERPENDICULARLY TO THE DIRECTION OF ITS TRANSLATION

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On solving exactly relativistic equations of motion for the model molecule of two particles coupled elastically, it has been shown that in the framework of relativistic mechanics this system, and in general any closed system of interacting particles, is not inertial. In particular, the translational velocity of the mass center of such a system has, as a consequence of the nonlinearity of the equations, oscillatory components reflecting its internal transverse oscillations — it is a pulsed motion. This effect can in principle be seen in the time-of-flight experiments. The force constant of elastic coupling in the system, as seen by the observer at rest, is shown to decline with increase of the total momentum of the system.

При точном решении релятивистских уравнений движения применительно к модели молекулы, состоящей из двух частиц, связанных упругими силами, показано, что с точки зрения релятивистской механики эта система и вообще какие-нибудь системы взаимодействующих частиц не являются инерциальными. В частности, скорость перемещения центра масс в такой системе имеет как следствие нелинейности уравнений осциллирующие компоненты, отражающие его внутренние поперечные осцилляции в виде пульсирующего движения. Этот эффект может, в принципе, проявиться во времяпролетном эксперименте. Показано, что силовая константа связи в системе, как представляется наблюдателю в состоянии покоя, уменьшается с ростом полного момента системы.

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Let us remind the basic identity for a free particle within the Special Relativity Theory (SRT). For the particle of the rest mass μ, energy E, and momentum $\mathbf{P}$, the following equation holds (in the units such that the velocity of light $c = 1$), see references [1–4]:

$$E^2 - \mathbf{P}^2 = \mu^2. \tag{1}$$

One should ask what happens, if the particle is a sort of «molecule» — it is composed of subparticles held together with some interparticle forces? From the point of view of an observer at rest, the following fundamental questions should be asked:

1. Do the internal motions of the component particles of the molecule have any impact on the movement of its center of mass?

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2. Can we treat such a molecule as a free particle of fixed mass?

3. Does the observed coupling between the component particles depend on the average velocity of the molecule?

In classical mechanics the answers are known — no, yes, and no, respectively [1]. But when the particles move fast, the case enters the SRT range and the equations of motion are nonlinear [2–4]. The answer is then neither obvious nor simple and there is a room for surprise. Even the very formulation of the equations of motion for many interacting particles in the framework of the SRT meets conceptual problems, because the energy-momentum 4-vectors are in general not additive [2, 3].

The questions are of fundamental character, so it makes sense to clear the situation using a simplest, but fully soluble model. Let us consider the movement of a cluster of two mutually coupled identical particles — we shall call it a molecule. It can be not only a typical chemical molecule like H_2 or O_2, but also a nucleus, like deuterium. We assume that no external forces act on the molecule. The particles have the mass M and are coupled with an elastic spring characterized by the force constant (Hooke modulus) k. We shall consider the «transverse» (T) configuration of motion of this «M_2 molecule»: its mass center moves along the x axis while the coupling acts along the y direction perpendicular to the x, and this is the direction of mutual oscillation of the particles (see Fig. 1, b). Due to symmetry of this T-configuration, it is natural to assume that the molecule exhibits the (1, –1) oscillation even at relativistic velocities of both particles.

Before taking on the relativistic problem of moving molecule, let us first refer to the old SRT problem of the oscillation period of a classic linear oscillator. Such a model — a particle coupled to a heavy wall with a spring, see Fig. 1, a, — has first been treated by Penfield and Zdesis (PZ), starting form the energy conservation law [5]. The solution for the oscillation amplitude vs. time is given in terms of elliptic integrals. At low velocity of the particle the movement is harmonic ($\sin(\omega t)$-like amplitude of oscillation vs. time). Moreau, Easther and Neutze [6] meticulously reconsidered the problem starting from the relativistic Lagrangian of Synge [7]. They emphasize the anharmonic oscillation aspects of the motion of relativistic oscillator and show that if the velocity of the particle tends to the velocity of light, the motion approaches a triangular (saw-like) pattern, with the particle velocities close to $\pm c$, alternatively. Young-Sea Huang [8] has derived the related equations of motion as a

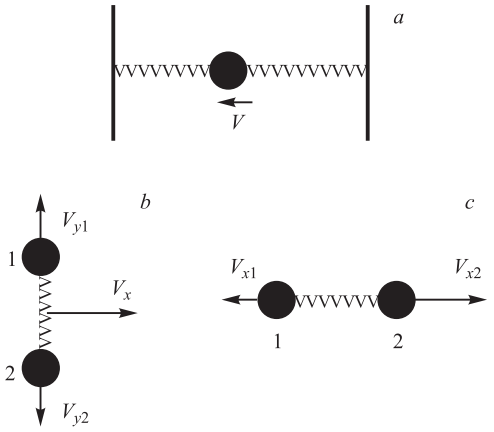

Fig. 1. Oscillating systems referred to: a) single particle coupled with elastic spring to immobile walls; b) the molecule M_2 of two particles coupled with elastic spring, executing the symmetric (1, –1) oscillation along the direction y (vertical), while moving as a whole horizontally (x direction) — the transverse mode T; c) the M_2 molecule oriented, oscillating and executing translation as a whole along the x direction — the longitudinal mode L

simple application of the Lorentz-covariant proper-time Lagrangian formulation of relativistic mechanics.

In the unpublished paper we have considered the relativistic problem of motion of two particles coupled elastically, starting form the postulate that the energy conservation law and the momentum conservation law are separately valid, see [9]. As shown in Fig. 1, *b*, the inner oscillation in the T-configuration has symmetry (1, –1) and the problem reduces here mathematically (however, with parameters depending on the total momentum of the molecule) to the Penfield–Zdetsis problem of the single-particle relativistic oscillation. The solution for the configuration L (see Fig. 1, *c*) has also been found, but it is mathematically more complex. A solution for arbitrary angle between the oscillation and the translation direction has not been found yet.

It is not a priori obvious that both conservation laws can be applied separately for this relativistic system. Therefore, in this paper we analyze the problem of motion of two coupled particles moving in the configuration T, starting from the beginning — from the full SRT formulation of the Newton equations of motion [1–4]. *The reason for such a choice is that for such symmetric transverse motion of this two-body system the SRT equations of motion can be solved exactly and the solution is transparent.* It is perhaps the simplest many-body system one can so conveniently solve within the SRT. We show below that for the T-mode the validity of two independent conservation laws is not a wishful postulate, but a direct consequence of the SRT equations of motion. It allows us to give doubtless answers to the questions set above.

In true physical molecule like H_2 or N_2 the coupling (bond) between the two nuclei is a consequence of electromagnetic forces between the nuclei and electrons, which are known to obey the FL transformation. It suggests solving the problem of fast-moving molecules from first principles. However, full ab initio quantum many-body theory of the coupling in such a molecule does not exist yet, although hopefully it will emerge in future. At present, in order to describe the mutual oscillation of the two nuclei, we introduce a spring-like coupling between them, characterized by the force constant k. Such a step should be sufficient at least for small oscillations — in the harmonic region of oscillations. One can expect that in the ab initio microscopic theory this quantity will be in future expressed in terms of its physical components — electron and nucleus masses and charges, Planck constant, light velocity. Here it is a phenomenological parameter, which possibly depends on the velocity v_x of translation of the molecule with respect to the observer. The approximate form of this dependence is derived below.

With the particles of the mass M, labeled 1 and 2, the SRT equations of the T-motion can be written in the textbook form $d/dt(\gamma \mathbf{v}) = \mathbf{F}$, see [1–4], where $\mathbf{F}$ is the Newtonian force (in general the 3D part of Minkowski's force 4-vector), $\mathbf{v}$ is the 3D particle velocity, and γ is the Fitzgerald–Lorentz (FL) time-delay factor. Here the movement of the molecule is assumed to occur in the $x-y$ plane and the equations look as follows:

$$M\frac{d}{dt}(\gamma v_{1y}) = -k(y_1 - y_2), \tag{2a}$$

$$M\frac{d}{dt}(\gamma v_{2y}) = -k(y_2 - y_1), \tag{2b}$$

$$M\frac{d}{dt}(\gamma v_x) = 0, \tag{2c}$$

where $v_{1y} = dy_1/dt$, $v_{2y} = dy_2/dt$, $v_x = v_{1x} = v_{2x}$, k is the force constant (Hooke's modulus) of the spring coupling of the particles and the FL factor is

$$\gamma = (1 - v^2)^{-1/2}, \tag{3}$$

where v is the velocity of the point-mass M seen by the observer at rest. Note that due to mirror symmetry of the molecule oscillation in the T-configuration we have for the oscillation amplitudes $y_1 = -y_2 = y$ and for the velocities $v_{1y} = -v_{2y} = v_y$, so one can write $v_{1y}^2 = v_{2y}^2 = v_y^2$, and $v^2 = v_x^2 + v_y^2$. Therefore, the dilatation factor for both particles is just single one. By this (1, –1) symmetry of the molecule oscillation in the T-configuration the two particles are equivalent. Effectively each particle moves in the gutter-pipe like field of potential $V(x, y) = ky^2$. The two-body problem formally turns here into the one-body problem. It follows that for this specific system the coupling retardation problems, due to a finite velocity of the position signal between interacting particles, do not appear. The equations (2a), (2b) for the oscillation amplitudes can be substituted by just one for the $y(t)$:

$$M\frac{d}{dt}(\gamma v_y) = -2ky. \tag{4}$$

Even without integration one can see that due to the occurrence of both components of the velocity in the γ factor, the x and y components of the particle movement cannot be separated. On integrating first Eq. (2c) we obtain the *momentum conservation law*

$$\frac{Mv_x}{\sqrt{1 - v_x^2 - v_y^2}} = \text{const} = \frac{P}{2}. \tag{5}$$

Immediately we get from it a relation between the velocity of inner oscillatory motion v_y and the velocity v_x of translational motion of the molecule as a whole

$$v_x^2 = \frac{P^2}{4M^2 + P^2}(1 - v_y^2). \tag{6}$$

It says that the spring-induced transverse oscillation of the molecule imposes an oscillatory component onto the translational motion velocity v_x of the center of mass of the molecule, see Fig. 1, *b*. In other words — as the equations of motion are nonlinear, the center of mass of oscillating molecule does not move uniformly, and its velocity shows *a pulsation*, although the molecule momentum P is constant in time. This is a purely relativistic effect, due to transfer of energy between both forms of motion. It should be possibly observed with flying nuclei rather than with chemical molecules, because only in nuclei one can expect the inner subparticle velocities to be sufficiently high. The effect is illustrated in Fig. 2.

One can argue that the spring coupling k of both particles of the molecule is a physical object and thus it is not quite safe to treat the force constant k as independent of the molecule velocity. It is true, but we can see that luckily this parameter does not appear at all in the momentum conservation law (5). It suggests that the impact of the inner oscillation on the translational motion of the molecule given by Eq. (6) is a real effect. The answer to the first question is therefore YES. We shall return to the velocity dependence of the k, as seen by the observer at rest, later on.

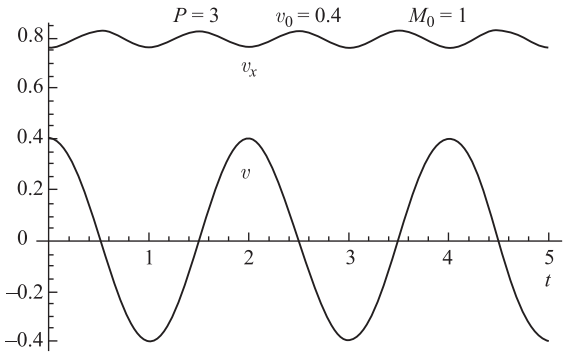

Fig. 2. Relativistic «pulsed» motion of the two-particle molecule M_2 in the mode T of Fig. 1, *b*. Picture shows the interrelation given by Eq. (6) between the molecule velocities vs. time. $M = M_0 = 1$ is the particle mass. Here v stays for the transverse oscillation velocity v_y, $v_0 = v_{y=0}$, while v_x is the translational velocity of the center of mass of the molecule. The molecule momentum given by Eq. (4) is assumed to be $P = 3$ (in the units $c = 1$)

Using Eq. (6), one can eliminate v_x from the dilatation factor and obtain such a factor for the y-motion alone:

$$\gamma_{=}\frac{\sqrt{1+P^2/(4M^2)}}{\sqrt{1-v_y^2}} \equiv \gamma_y. \tag{7}$$

On inserting it into Eq. (4) we have

$$\frac{d}{dt}(\gamma_y v_y) = -\frac{2k}{M}y. \tag{4′}$$

Note that for $\gamma_y \to 1$ (low velocity v_y) it is the classic Newton equation for the harmonic oscillator with the frequency $\omega^2 = 2k/M$. On multiplying both sides with the $v_y = dy/dt$ we can do the integration and obtain

$$M\left(1+\frac{P^2}{4M^2}\right)^{1/2}(1-v_y^2)^{-1/2} + ky^2 = \mathrm{const} = \frac{E}{2}. \tag{8}$$

In this way we have arrived at the *energy conservation law* for the T-motion. Mathematically it is the Penfield–Zdetsis equation [5], with the particle mass M multiplied by the factor $(1+P^2/(4M^2))^{1/2}$. In the preliminary paper [9] we have derived this equation directly from the postulates of energy and momentum conservation. The oscillation amplitude y vs. time t plots for several initial velocities of oscillation and translation momentum P values have been evaluated and shown there. Here let us only address the matter of the «rest mass» of such a complex particle as the «molecule M_2».

At the center of oscillatory motion there is $y = 0$ and the inner velocity v_y is the highest. We shall call it the «initial» velocity v_0. Equation (8) allows us to connect the energy E, the momentum P and the v_0:

$$E = 2M\left(1+\frac{P^2}{4M^2}\right)^{1/2}(1-v_0^2)^{-1/2}. \tag{9}$$

On inserting it into Eq. (1) we get for the «rest mass» of the molecule M_2 executing its T-motion

$$\mu^2 = \frac{4M^2 + v_0^2 P^2}{1 - v_0^2} = (2M)^2 + \frac{4M^2 v_0^2}{1 - v_0^2} + \frac{v_0^2 P^2}{1 - v_0^2}. \tag{10}$$

We can see that if there is an inner oscillation characterized by the initial velocity v_0, *the μ differs from the sum of rest masses*, which is equal to $2M$, not only by the «mass defect» (second term in the RHS), which depends only on the inner dynamics of the molecule and thus can be incorporated into a mass of free particle. There is also the third term in the RHS, which depends on the total momentum P of the molecule combined with the inner velocity v_0. *Therefore, the μ^2 is not a constant mass independent of the translational motion of this system*. This sets the answer to the second question asked at the beginning of this paper — in spite of the fact that the force here is only an inner force, the molecule M_2 cannot be treated as the free particle. *This statement should be immediately generalized — no complex particle having inner dynamics is a free particle in the sense of Eq. (1). It means that in principle no real complex system, including atoms and nuclei, is the inertial system. Practically, it often does, because the corrections are only of the order $v_y^2 P^2$.*

Only if the molecule does not move, $P = 0$, we can assign to it a well-defined mass

$$\mu_0^2 \equiv E_{P=0}^2 = \frac{4M^2}{1 - v_0^2} = (2M)^2 + \frac{4M^2 v_0^2}{1 - v_0^2}. \tag{11}$$

One can see that this mass (energy) is larger than $2M$ not only by the well-known kinetic-energy-like contribution due to the inner motion — it is additionally enhanced via the relativistic denominator involving the «initial» velocity $v_0 \equiv v_y$ at $y = 0$. It follows that the binding energy B in this model is

$$\frac{B}{2M} = \frac{1}{2} \frac{v_0^2}{1 - v_0^2}. \tag{12}$$

Equation (8) allows us to determine the dependence of the force constant k on the average translational velocity v_x of the molecule, at least for low inner velocity $v_0 \ll 1$. Let us evaluate the oscillation period appearing within the present model. First, we ask for the velocity v_0 at $y = 0$. Equation (8) gives, with $A = 2M(1 + P^2/4/M^2)^{1/2}$,

$$v_0 = \left(1 - \left(\frac{A}{E}\right)^2\right)^{1/2}. \tag{13}$$

Then let us introduce the notion of maximum oscillation amplitude y_m as such that $v_y = 0$. Equation (8) gives

$$y_m = \sqrt{\frac{E - A}{2k}}. \tag{14}$$

At $v_0 \ll 1$ the oscillation is harmonic, $y(t) = y_m \sin(2\pi t/T)$ and $v_0 = y'(0)$, thus

$$T = 2\pi \frac{y_m}{v_0} = 2\pi \sqrt{\frac{E}{2k(1 + A/E)}}. \tag{15}$$

At $v_0 \to 0$ we have $E \cong A$ and within the present model the oscillation period is

$$T = 2\pi\sqrt{\frac{M}{2k}}\left(1 + \frac{P^2}{4M^2}\right)^{1/4}. \tag{16}$$

We are now in a position to look at the problem of oscillation period of the moving molecule straight from the point of view of the FL time dilatation. To do it, we first introduce the force constant k_0 of the molecule *at rest* executing *small* oscillations. Equations of motion are such as Eqs. (2a)–(2c) with $\gamma = 1$ and k_0 instead of k. The corresponding period of oscillation is $T_0 = 2\pi\sqrt{M/2k_0}$ (see, e.g., [10]). We know a priori that the period of a physical system running with the constant velocity v_x with respect to the observer at rest is given by the FL time-dilatation formula

$$T = T_0(1 - v_x^2)^{-1/2}. \tag{17}$$

The difference between Eqs. (16) and (17) should be attributed to the fact that so far we have treated the spring constant as the quantity independent of the motion of molecule with respect to the observer. In fact, the interparticle coupling is always realized by physical particles and fields, so their transformation properties should be accounted for. The present derivation helps one to establish the approximate form of such a transformation for the force constant k.

To avoid, at this point, conceptual problems related to the oscillatory component of the translational velocity, discussed above, we assume that the v_x is here the «average» translational velocity of the M_2 molecule following from Eq. (5) with $v_y \to 0$. To reconcile Eqs. (16) and (17), we have to assume the following relation between k and k_0:

$$k = (1 - v_x^2)\left(1 + \frac{P^2}{4M^2}\right)^{1/2} k_0. \tag{18}$$

It is the phenomenological derivation of the transformation of the spring elastic field characterized by the force constant k, valid for the moving molecule at low transversal velocity $v_y \to 0$. Using again this condition in Eq. (5), we can rewrite Eq. (18) in even simpler form

$$k \cong \left(1 + \frac{P^2}{4M^2}\right)^{-1/2} k_0. \tag{19}$$

The physical content of these equations is that for the observer at rest the elastic coupling k between the two particles oscillating within the running molecule is *weaker* than the same coupling in this molecule at rest. This is the answer to the third question set in this paper. The equations give the transformation law for the force constant k, valid at least for relatively low momentum P or translational velocity v_x of the molecule.

In principle, one could try to write down such a transformation law for arbitrary initial velocity, starting from the exact solution for the oscillation period T given in [9]. It is a rather complex task and far from being transparent, due to its formulation in terms of elliptic integrals.

One can expect observing the pulsed motion of fast-running molecules either produced in accelerators (molecular ions) or in the spectra of molecules accelerated by gravitation in cosmic space. As the case for experimental verification we can take deuterium D. The masses of both component nucleons are almost equal and we have $2M = M_p + M_n = 1877.852$ MeV, while

the binding energy $B = 2.225$ MeV, see [10]. Thus, one obtains by Eq. (12), as the highest velocity of relative oscillation of proton against neutron in deuterium, the approximate value $v_0 = 0.0486$ (in the units c). The amplitude of oscillatory component v_y^2 in the translational velocity, Eq. (5), is of the order 0.002 — small but reasonable. However, if we take two deuterium radii $2R = 8.6$ fm as the amplitude y_m of relative proton–neutron oscillation, the inner velocity v_0 as above, and the oscillation period given by the LHS of Eq. (15), then we obtain as a reasonable estimate for the period of inner oscillation in deuterium a very small number $T = 3.7 \cdot 10^{-21}$ s, rather difficult to experimentally deal with.

For chemical molecules like N_2 or O_2, the oscillation period is of the order 10^{-12} s, see, e.g., [11], but the relative atom–atom velocity is only of the order 1 km/s, so the component v_y^2 in Eq. (6) is as small as 10^{-10}. It follows that the experimental determination of the relativistic pulsation of the velocity of complex bodies, predicted here, needs a special care.

SUMMARY

We have shown that inner motions of subparticles in a moving physical system manifest themselves in making the system's center-of-mass translational movement nonuniform — such a multiparticle system is never fully inertial. This is a pulsed motion and in principle can be observed in time-of-flight experiments. The observed coupling between oscillating subparticles in such a running molecule is weaker than in the molecule at rest.

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