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FLUCTUATIONS AND CORRELATIONS IN PION SYSTEM WITH FIXED ISOSPIN

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The statistical system of π^0, π^+ and π^- mesons with zero total isospin is studied. For neutral pions there is the enhancement of the fluctuations, whereas for charged pions the isospin conservation suppresses fluctuations. The correlations between the numbers of charged and neutral pions are observed for finite systems.

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INTRODUCTION

The aim of the present talk[1] is to study some aspects of non-Abelian symmetries in the statistical models. We consider $SU(2)$-isospin symmetry group for the pion system. The role of the isospin conservation in a many-body system was first considered in the pioneering paper of Bethe [2]. Many efforts were then aimed at studies of the pion system with fixed isospin [3]. An effective theoretical formalism for non-Abelian symmetries in the statistical mechanics was developed in [4] on the basis of the group projection technique. It allowed one to consider the impact of the isospin conservation on the particle abundances and the form of their momentum spectra in the statistical models of hadron production [5]. The group projection technique was also used to calculate the colorless partition function of the quark–gluon gas with $SU(N_c)$-color symmetry [6].

Our primary interest is to study an influence of non-Abelian charge conservation on the particle number fluctuations. It was recently found [7] that exact conservation of Abelian (additive) charges causes the suppression of the particle number fluctuations. In the present study we restrict our consideration to the simplest statistical system with non-Abelian symmetry — an ideal pion gas with zero isospin $I = 0$. Most discussions are done within Boltzmann statistics. This makes it possible to obtain transparent analytical results and compare them with those in the canonical ensemble for zero electric charge $Q = 0$.

1. PARTITION FUNCTION

The partition function of the ideal Boltzmann gas of pions π^+, π^-, π^0 in the grand canonical ensemble (GCE) reads

$$Z_{\rm GCE} = \sum_{N_0,N_+,N_-=0}^{\infty} \frac{(\lambda_0\; z)^{N_0}}{N_0!}\,\frac{(\lambda_+\; z)^{N_+}}{N_+!}\,\frac{(\lambda_-\; z)^{N_-}}{N_-!} = \exp\left[(\lambda_0+\lambda_++\lambda_-)\, z\right], \tag{1}$$

[1] See also the paper [1] for more details.

where z is the one-particle partition function $z = V/(2\pi^2) \int_0^\infty p^2 dp \exp\left(-\frac{\sqrt{p^2+m^2}}{T}\right) = V/(2\pi^2)\, Tm^2\, K_2\,(m/T)$. Here V and T are the system volume and temperature, m is the pion mass[1], and K_2 is the modified Hankel function. The auxiliary parameters λ_j with $j = 0, +, -$ are introduced to calculate the mean pion multiplicities, fluctuations and correlations. We take $\lambda_j \equiv 1$ in the final formulae.

In the case of exact charge conservation, i.e., in the canonical ensemble (CE) with zero charge $Q = 0$, the partition function is (see, e.g., [7, 8])

$$Z_{Q=0} = \sum_{N_0, N_+, N_- = 0}^{\infty} \delta\left(N_+ - N_-\right) \frac{\left(\lambda_0 z\right)^{N_0}}{N_0!} \frac{\left(\lambda_+ z\right)^{N_+}}{N_+!} \frac{\left(\lambda_- z\right)^{N_-}}{N_-!} =$$

$$= \exp\left(\lambda_0 z\right) \frac{1}{2\pi} \int_0^{2\pi} d\phi \, \exp\left\{z\left(\lambda_+ \exp\left[i\phi\right] + \lambda_- \exp\left[-i\phi\right]\right)\right\}. \quad (2)$$

Note that an exact charge conservation in the CE (2) does not affect the neutral pions. Their number distribution remains the Poissonian one, the same as in the GCE (1).

The partition function with total isospin $I = 0$ can be obtained using group projection technique. Pions are transformed under vector (adjoint) representation of the $SU(2)$ group. This group has three parameters which can be chosen as Euler angles $\boldsymbol{\alpha} = \alpha, \beta, \gamma$. In this case the diagonal matrix elements have the following form [9]:

$$D^1_{\pm 1, \pm 1}(\alpha, \beta, \gamma) = \mathrm{e}^{\pm i(\alpha+\gamma)} \left(\frac{1 + \cos\left(\beta\right)}{2}\right), \quad D^1_{0,0}(\alpha, \beta, \gamma) = \cos\left(\beta\right). \quad (3)$$

The partition function is then presented as [10]

$$Z_{I=0} \int d\mu \sum_{N_0, N_+, N_- = 0}^{\infty} \frac{\left[\lambda_0 z D^1_{0,0}(\boldsymbol{\alpha})\right]^{N_0}}{N_0!} \frac{\left[\lambda_+ z D^1_{1,1}(\boldsymbol{\alpha})\right]^{N_+}}{N_+!} \frac{\left[\lambda_- z D^1_{-1,-1}(\boldsymbol{\alpha})\right]^{N_-}}{N_-!} =$$

$$= \int d\mu \, \exp\left[\lambda_0 z D^1_{0,0}(\boldsymbol{\alpha}) + \lambda_+ z D^1_{1,1}(\boldsymbol{\alpha}) + \lambda_- z D^1_{-1,-1}(\boldsymbol{\alpha})\right]. \quad (4)$$

Substituting explicit expressions for the Haar group measure $d\mu$ and matrix elements $D^t_{t_3, t_3}$ (3) in Eq. (4), changing variables $\phi = \alpha + \gamma$, $\varphi = (\alpha - \gamma)/2$, $\cos\left(\beta\right) = x$ and integration over φ, one obtains

$$Z_{I=0} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} dx \, \exp\left[\lambda_0 z x + z \frac{1+x}{2} (\lambda_+ \mathrm{e}^{i\phi} + \lambda_- \mathrm{e}^{-i\phi})\right]. \quad (5)$$

Comparing $Z_{I=0}$ (5) with the partition function $Z_{Q=0}$ (2), one observes an additional x-integration in Eq. (5). It reflects a presence of the particle number correlations between neutral and charged pions which were absent in the GCE and CE.

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[1] We neglect a small difference between the masses of charged and neutral pions.

2. FLUCTUATIONS AND CORRELATIONS OF π^0, π^+, π^-

To calculate the mean multiplicities, correlations, and fluctuations for neutral and charged pions, one has to return back to presentations of the partition functions by Eqs. (1), (2), and (5). One finds

$$\langle N_j \rangle = \frac{1}{Z}\frac{\partial Z}{\partial \lambda_j}\bigg|_{\boldsymbol{\lambda}=1}, \quad \langle N_i\, N_j \rangle \equiv \frac{1}{Z}\frac{\partial}{\partial \lambda_i}\left(\lambda_j \frac{\partial Z}{\partial \lambda_j}\right)\bigg|_{\boldsymbol{\lambda}=1}. \tag{6}$$

Using Eq. (6), one obtains for the mean multiplicities of neutral and charged particles. The ratios $R_0 = \langle N_0 \rangle / z$, $R_\pm = \langle N_\pm \rangle / z$ are shown in Fig. 1, *a* for $Q = 0$ and $I = 0$ statistical ensembles.

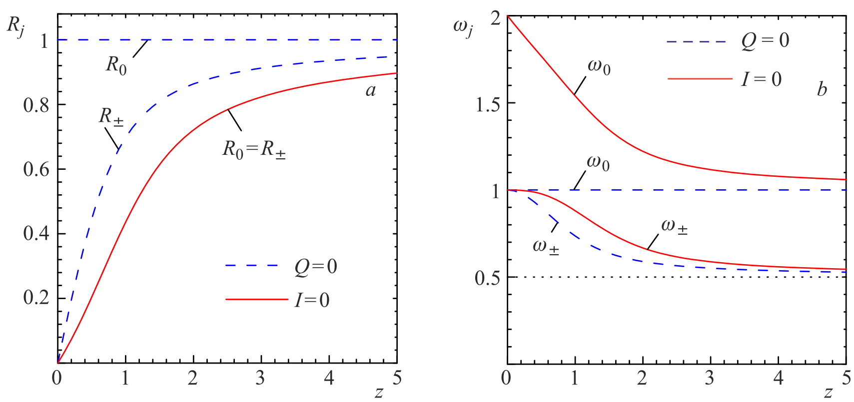

Fig. 1. The ratios R_0 and $R_\pm$ (*a*) and the scaled variances ω_0 and $\omega_\pm$ (*b*) in the CE with $Q = 0$ (dashed lines) and in the ensemble with $I = 0$ (solid lines)

Note that $R_0 = R_\pm = 1$ in the GCE. In the CE, $R_0^Q = 1$; i.e., the mean number of neutral pions is not affected by the charge conservation law. The mean number of charged pions is suppressed in the CE, $R_\pm^Q < 1$ [8]. The behavior in the statistical ensemble with $I = 0$ differs from that in the CE with $Q = 0$. The pion mean numbers are the same for all charge pion states π^0, π^+, and π^-. The suppression of these pion multiplicities at $I = 0$ is stronger than that for π^- or π^+ in the CE with $Q = 0$: $R_0^I = R_\pm^I < R_\pm^Q < R_0^Q = 1$.

One can see that $R_j \to 1$ at $z \to \infty$ for all $j = 0, +, -$ in both $Q = 0$ and $I = 0$ statistical ensembles. Thus, the suppression of the mean numbers of charged and neutral pions is the finite volume effect. At small z: $R_\pm^Q \cong z$, $R_0^I = R_\pm^I \cong z/3$.

The second derivatives in Eq. (6) can be calculated using the partition functions (1), (2), and (5) for different statistical ensembles. The scaled variances ω_j and correlation coefficients ρ_{ij},

$$\omega_j \equiv \frac{\langle N_j^2 \rangle - \langle N_j \rangle^2}{\langle N_j \rangle}, \quad \rho_{ij} \equiv \frac{\langle N_i N_j \rangle - \langle N_i \rangle \langle N_j \rangle}{\sqrt{\omega_i \omega_j \langle N_i \rangle \langle N_j \rangle}}, \tag{7}$$

define the main characteristics of the pion multiplicity distributions. As seen from Fig. 1, the isospin conservation with $I = 0$ gives the same pion number fluctuations as the CE with $Q = 0$, $\omega_0^I \to 1$ and $\omega_\pm^I \to 1/2$, at $z \to \infty$.

However, the results for finite systems are rather different. In the CE with $Q = 0$ the fluctuations of neutral particles are the same as in the GCE, $\omega_0^Q = 1$. The scaled variance $\omega_\pm^Q$ in the CE with $Q = 0$ was calculated in [7]. At $z \ll 1$ one finds: $\omega_\pm^Q \cong 1 - z^2/2$ and $\omega_0^I \cong 2 - z/2$, $\omega_\pm^I \cong 1 - z^2/30$.

There are no correlations between the numbers of π^0, π^+, π^- in the GCE. All correlation coefficients defined by Eq. (7) are equal to zero, $\rho_{0+} = \rho_{0-} = \rho_{+-} = 0$. The charge is exactly conserved in the $Q = 0$ and $I = 0$ statistical ensembles. This brings the strongest correlations between the numbers of π^+ and π^-, i.e., $\rho_{+-} = 1$, and this means equal numbers N_+ and N_- in each microscopic state of the system. The correlations between the numbers of π^0 and $\pi^\pm$ are absent in the CE with $Q = 0$, but exist in the statistical ensemble with $I = 0$. The correlation coefficient $\rho_{0\pm}^I$ at $I = 0$ has the maximal value $\rho_{0\pm}^I \approx 0.19$ at $z \approx 1$ and goes to zero at $z \to \infty$.

Due to the exact charge conservation, negative and positive pions may appear only as $\pi^+\pi^+$ pairs. For $N_{\rm ch} \equiv N_+ + N_-$, the scaled variance is then two times larger than that for positive (negative) pions $\omega_{\rm ch}^Q = 2\omega_-^Q = 2\omega_+^Q$. The same relation, $\omega_{\rm ch} = 2\omega_\pm$, is also valid at $I = 0$.

The role of Bose effects for the fluctuations in the CE was considered in [11]. A generalization of Eq. (5) for Bose statistics gives the following expression:

$$Z_{I=0}^{\rm Bose} = \frac{1}{8\pi^2} \int\limits_0^{2\pi} d\alpha \int\limits_0^{2\pi} d\gamma \int\limits_0^{\pi} d\beta \sin\beta \times \\ \times \exp\left[\sum_{n=1}^{\infty} \frac{z_n}{n} \left(\lambda_0^n \cos^n\beta + \left(\frac{1+\cos\beta}{2}\right)^n \left(\lambda_+^n {\rm e}^{in(\alpha+\gamma)} + \lambda_-^n {\rm e}^{-in(\alpha+\gamma)}\right)\right)\right], \quad (8)$$

where $z_n = VTm^2 K_2\left(nm/T\right)/(2\pi^2 n)$. The Boltzmann approximation corresponds to the first term $n = 1$ in the sum in Eq. (8). In the case of quantum statistics, the partition function depends not only on the one particle partition function z, but additionally on the value of m/T. Bose statistics makes the pion number fluctuations larger. These effects are always stronger for smaller values of the m/T ratio. For $m/T \to 0$ we find $\omega_0^I \cong 2.26$ at $z \to 0$, and $\omega_0^I \cong 1.368$ at $z \to \infty$. The corresponding results for the charged particles are the following: $\omega_\pm^I = 1$ at $z \to 0$, and $\omega_\pm^I \cong 0.684$ at $z \to \infty$. The results for $z \to \infty$ coincide with those in the canonical ensemble with $Q = 0$.

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