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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

TOWARDS SOLUTION OF SUPERSTRING THEORY IN $AdS_3 \times S^3$ WITH MIXED FLUX

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We review recent work on S-matrix of type IIB string in $AdS_3 \times S^3 \times T^4$ background with mixed RR and NSNS 3-form fluxes. This worldsheet theory is expected to be integrable, opening the possibility of computing its exact spectrum for any values of the coefficient q of the NSNS flux and the string tension.

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Below, we review the following [1, 2] recent study of S-matrix for elementary massive excitations of superstring theory on $AdS_3 \times S^3 \times T^4$ with mixed RR + NSNS 3-form flux parameterized by $q \in (0, 1)$. This model interpolates between the purely RR flux case ($q = 0$), described by a supercoset GS superstring, and the purely NSNS flux case ($q = 1$), which can be described by a supersymmetric WZW model. Its classical integrability [3] is expected to extend to the full quantum level and thus, like in the $AdS_5 \times S^5$ case [4], should allow for an exact solution for the string spectrum for any value of the string tension h and the parameter q. This should shed light on the corresponding dual 2-d CFT, which is currently not understood beyond its supersymmetry-protected BPS sector.

The tree-level ($h \to \infty$) S-matrix was found in [1] from the superstring action and it was extended in [2] to the exact in h result using symmetry algebra considerations and generalizing the pure RR ($q = 0$) result of [5]. A key idea is that while the superstring symmetry group and the symmetry algebra of the S-matrix do not depend on q, the representation of the latter on particle states does. This leads to a q-modified exact «magnon» dispersion relation. In the $q = 0$ case [5], the exact S-matrix (written in terms of the Zhukovsky variables $x^{\pm}$) is completely fixed, up to two phases, by its symmetry algebra and satisfies the Yang–Baxter equation without the need to use explicit form of the dispersion relation. To generalize to the $q \neq 0$ case, we need a new set of the Zhukovsky variables $x^{\pm}_{\pm}$ (the $_{\pm}$ subscripts correspond to positively/negatively charged states) that are consistent with the q-modified dispersion relation and in terms of which the representation parameters and the exact S-matrix take the same form as in the $q = 0$ case. The S-matrix satisfies the Yang–Baxter equation and involves four phases that remain to be determined by additional considerations.

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The tree-level S-matrix for the massive BMN modes of the superstring theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux was found in [1]. The flux is parameterized by $0 \leqslant q \leqslant 1$, with $q = 0$ ($\hat{q} = 1$) corresponding to the pure RR case and $q = 1$ ($\hat{q} = 0$) — to the pure NSNS case,

$$q^2 + \hat{q}^2 = 1, \quad \hat{q} = \sqrt{1 - q^2}. \tag{1}$$

The second parameter is the string tension h related to the radius of AdS_3 or S^3:

$$h = \frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}. \tag{2}$$

The quantized coefficient of the WZ term in the action is related to q and h as $q\sqrt{\lambda} = k$, i.e., $k = 2\pi hq$. The near-BMN expansion of the $AdS_3 \times S^3 \times T^4$ superstring action describes $4 + 4$ (bosonic + fermionic) modes with mass $\hat{q} = \sqrt{1 - q^2}$, and $4 + 4$ massless modes. The corresponding S-matrix for the massive modes can be written as the graded tensor product of two copies of an S-matrix describing the scattering of $2 + 2$ massive modes. Let us denote the two massive bosons associated to the orthogonal directions of S^3 as the complex scalar $y = y_1 + iy_2$ and the corresponding scalar for AdS_3 as $z = z_1 + iz_2$. The four massive fermions will be represented as two complex Grassmann fields ζ and χ. The factorization property means that the S-matrix for $\{y, z, \zeta, \chi\}$ can be constructed from an S-matrix for $\{\phi, \psi\}$, which takes the following form:

$$\begin{aligned}
&\mathbb{S}\left|\phi_\pm\phi'_\pm\right\rangle = A_\pm L_{1\pm}\left|\phi_\pm\phi'_\pm\right\rangle, &&\mathbb{S}\left|\phi_\pm\psi'_\pm\right\rangle = A_\pm L_{3\pm}\left|\phi_\pm\psi'_\pm\right\rangle + A_\pm L_{5\pm}\left|\psi_\pm\phi'_\pm\right\rangle,\\
&\mathbb{S}\left|\psi_\pm\psi'_\pm\right\rangle = A_\pm \Lambda_{1\pm}\left|\psi_\pm\psi'_\pm\right\rangle, &&\mathbb{S}\left|\psi_\pm\phi'_\pm\right\rangle = A_\pm \Lambda_{3\pm}\left|\psi_\pm\phi'_\pm\right\rangle + A_\pm \Lambda_{5\pm}\left|\phi_\pm\psi'_\pm\right\rangle,\\
&\mathbb{S}\left|\phi_\pm\psi'_\mp\right\rangle = \bar{A}_\pm L_{6\pm}\left|\phi_\pm\psi'_\mp\right\rangle, &&\mathbb{S}\left|\phi_\pm\phi'_\mp\right\rangle = \bar{A}_\pm L_{2\pm}\left|\phi_\pm\phi'_\mp\right\rangle + \bar{A}_\pm L_{4\pm}\left|\psi_\pm\psi'_\mp\right\rangle,\\
&\mathbb{S}\left|\psi_\pm\phi'_\mp\right\rangle = \bar{A}_\pm \Lambda_{6\pm}\left|\psi_\pm\phi'_\mp\right\rangle, &&\mathbb{S}\left|\psi_\pm\psi'_\mp\right\rangle = \bar{A}_\pm \Lambda_{2\pm}\left|\psi_\pm\psi'_\mp\right\rangle + \bar{A}_\pm \Lambda_{4\pm}\left|\phi_\pm\phi'_\mp\right\rangle.
\end{aligned} \tag{3}$$

The structure of this S-matrix (3) is constrained by the requirement of a $U(1)^2$ symmetry under which $\{\phi, \psi\}$ have charges $\{1, 0\}$ and $\{0, 1\}$, respectively. The leading-order term in the expansion in the inverse string tension h^{-1} gives the tree-level S-matrix parameterized by ($L_{1\pm} = 1 + \frac{i}{2h} l_{1\pm} + \mathcal{O}(h^{-2})$, etc., see [1]):

$$\begin{aligned}
l_{1\pm} &= \frac{(p + p')(e'_\pm p + e_\pm p')}{2(p - p')}, \quad l_{2\pm} = \frac{(p - p')(e'_\mp p + e_\pm p')}{2(p + p')},\\
l_{3\pm} &= -\frac{1}{2}(e'_\pm p + e_\pm p'),\\
l_{4\pm} &= -\frac{pp'}{2(p + p')}\left[\sqrt{(e_\pm + p \pm q)(e'_\mp + p' \mp q)} - \sqrt{(e_\pm - p \mp q)(e'_\mp - p' \pm q)}\right],\\
l_{5\pm} &= -\frac{pp'}{2(p - p')}\left[\sqrt{(e_\pm + p \pm q)(e'_\pm + p' \pm q)} + \sqrt{(e_\pm - p \mp q)(e'_\pm - p' \mp q)}\right],
\end{aligned} \tag{4}$$

$$e_\pm = \sqrt{\hat{q}^2 + (p \pm q)^2}, \quad e'_\pm = \sqrt{\hat{q}^2 + (p' \pm q)^2}. \tag{5}$$

Equation (5) gives the dispersion relation (which is the same for the bosonic and fermionic modes), $e^2_\pm = 1 + p^2 \pm 2q\,p$, generalizing the familiar BMN massive relativistic dispersion relation. The energy is minimized when $p = \mp q$ so that $\hat{q}$ is the mass of the corresponding excitations.

The type IIB supergravity background corresponding to the superstring under consideration is the near-horizon limit of the nonthreshold BPS bound state of NS5-NS1 and D5-D1 branes and can thus be obtained, e.g., by applying S-duality to the NS5-NS1 ($q = 1$) or D5-D1 ($q = 0$) solution. This means that the space-time symmetry of this background cannot depend on q. Indeed, the nontrivial "massive" $AdS_3 \times S^3$ part of the superstring action can be described by the same supercoset geometry $[PSU(1,1|2) \times PSU(1,1|2)]/[SU(1,1) \times SU(2)]$ [6] with q appearing only as a parameter in the action [3]. The algebra is

$$[\mathfrak{u}(1) \Subset \mathfrak{psu}(1|1)^2]^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3. \tag{6}$$

It is a subalgebra of the familiar $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$ which was a factor symmetry in the corresponding construction of the S-matrix of the $AdS_5 \times S^5$ superstring theory. For the supersymmetry algebra to close the parameters of representation should satisfy

$$a_\pm b_\pm = P_\pm, \quad c_\pm d_\pm = K_\pm, \quad a_\pm d_\pm = C_\pm + \frac{M_\pm}{2}, \quad b_\pm c_\pm = C_\pm - \frac{M_\pm}{2}. \tag{7}$$

These can easily be seen to imply that $C_\pm^2 = M_\pm^2/4 + P_\pm K_\pm$.

The factorized tree-level S-matrix of the theory with mixed 3-form flux ($q \neq 0$) co-commutes ($\Delta^{op}(\mathfrak{J})\mathbb{S} = \mathbb{S}\Delta(\mathfrak{J})$) with the supersymmetry algebra if the representation parameters have the following form at the leading order in the large h (near-BMN) expansion:

$$\begin{gathered}
a_\pm = \frac{\mathrm{e}^{-\frac{i\pi}{4}}}{\sqrt{2}} \sqrt{e_\pm + 1 \pm qp}, \quad b_\pm = -\frac{i\,\mathrm{e}^{\frac{i\pi}{4}}}{\sqrt{2}} \frac{\hat{q}p}{\sqrt{e_\pm + 1 \pm qp}}, \\
c_\pm = \frac{i\,\mathrm{e}^{-\frac{i\pi}{4}}}{\sqrt{2}} \frac{\hat{q}p}{\sqrt{e_\pm + 1 \pm qp}}, \quad d_\pm = \frac{\mathrm{e}^{\frac{i\pi}{4}}}{\sqrt{2}} \sqrt{e_\pm + 1 \pm qp}, \\
U_\pm = 1 + \frac{ip}{2h}, \quad M_\pm = 1 \pm qp, \quad C_\pm = \frac{e_\pm}{2}, \quad P_\pm = -\frac{i}{2}\,\hat{q}p, \quad K_\pm = \frac{i}{2}\,\hat{q}p.
\end{gathered} \tag{8}$$

$C_\pm$ thus plays the role of the energy. In the $q \to 0$ limit $P_\pm$ and $K_\pm$ are proportional to the spatial momentum and $M_\pm$ is the effective mass parameter, while in the $q \to 1$ limit $P_\pm$ and $K_\pm$ vanish, while $M_\pm$ is the spatial momentum shifted by ± 1. This leads to the following exact dispersion relation [1]:

$$e_\pm = \sqrt{1 \pm \frac{2k}{\pi} \sin \frac{p}{2} + 4h^2 \sin^2 \frac{p}{2}}. \tag{9}$$

As there is little solid knowledge about the corresponding dual 2-d CFT (beyond the supersymmetry protected BPS states and moduli space), it is hard to comment on the possible meaning of (9) in the small string tension or weak coupling region $h \to 0$. In general, the identification of the parameter h with the string tension $\sqrt{\lambda}/2\pi$ in (2) may be true only in the strong-coupling limit $\sqrt{\lambda} \gg 1$, i.e., h may be a nontrivial function of λ. This finite renormalization appears to be absent in the pure RR case of $q = 0$, and it should also be absent in the pure NSNS case of $q = 1$, when h is directly related to the integer level k. However, it may be present for a generic value of q. Indeed, there is a 1-loop shift in $h(\lambda)$ in the case of another 1-parameter deformation of the $AdS_3 \times S^3 \times T^4$ theory — the $AdS_3 \times S^3 \times S^3 \times S^1$ theory. It would be important to investigate this by a direct 1-loop superstring computation for $q \neq 0$.

Let us note that from the worldsheet sigma model point of view, the scattering of states with nontrivial dispersion relation (9) corresponds to the scattering of solitonic "giant-magnons", which may be viewed as elementary massive light-cone gauge quanta (usual BMN "magnons") "dressed" by quantum corrections to all orders in the h^{-1} expansion. The fact that the exact S-matrix of the elementary excitations with the standard quadratic relativistic dispersion relation can be rewritten as an S-matrix for the scattering of such "dressed" states with the dispersion relation (9) is, of course, a nontrivial consequence of the integrability of the model.

Let us now mention that the exact dispersion relation (9) corresponds to a discretization (in the spatial worldsheet direction) of the second-order differential operator appearing in the quadratic part of the $q \neq 0$ $AdS_3 \times S^3$ string action expanded near the BMN geodesic (or, equivalently, written in the BMN light-cone gauge). It is thus a natural 1-d lattice (or "spin chain") analog of the BMN dispersion relation. The quadratic term in the bosonic string action expanded near the BMN geodesic has the following form:

$$\begin{aligned} I &= \frac{1}{2}h \int d\tau\, d\sigma(-\partial^a y_r\, \partial_a y_r - y_r y_r + q\epsilon_{rs} y_r\, \partial_1 y_s) \\ &= \frac{1}{2}h \int d\tau\, d\sigma(\dot{y}_r^2 - y_r'^2 - y_r^2 + 2q y_1 y_2'). \end{aligned} \tag{10}$$

Here, the q-dependent term originates from the WZ term or the B-field coupling. y_r $(r, s = 1, 2)$ are two real scalars representing the transverse fluctuations in S^3. The same action is found for the two scalars z_r representing the transverse fluctuations in AdS_3. The massive fermionic modes have (after "squaring") an equivalent kinetic operator, albeit with the mass term $\sim \hat{q}$ originating not from the curvature as for the bosons, but rather from the RR flux coupling. Let us now assume that the spatial direction σ is compact with length $\ell = 2\pi\mathcal{J}$ (we rescale τ and σ by the semiclassical S^3 angular momentum or rotation frequency parameter $\mathcal{J}$). Furthermore, let us also discretize σ into J points with step ε,

$$\varepsilon = \frac{\ell}{J} = 2\pi\frac{\mathcal{J}}{J}, \quad y_{r(n)}(\tau) = y_r(\tau, n\varepsilon), \quad n = 0, \ldots, J-1, \quad y_{r(J)} = y_{r(0)}. \tag{11}$$

Assuming that the spatial derivative is defined as $y_r' \to \varepsilon^{-1}(y_{r(n+1)} - y_{r(n)})$, the discrete version of the action (10) becomes

$$I = \frac{1}{2}h\varepsilon \sum_{n=1}^{J} \int d\tau \left[\dot{y}_{r(n)}^2 - \varepsilon^{-2}(y_{r(n+1)} - y_{r(n)})^2 - y_{r(n)}^2 + 2q\, y_{1(n)}(y_{2(n+1)} - y_{2(n)})\right]. \tag{12}$$

The corresponding dispersion relation is $(e^2 - 1 - 4\varepsilon^{-2}\sin^2 p/2)^2 - 16q^2\varepsilon^{-2}\sin^2 p/2 = 0$, or

$$e_\pm^2 = 1 + 4\varepsilon^{-2}\sin^2\frac{p}{2} \pm 4q\varepsilon^{-1}\sin\frac{p}{2}, \tag{13}$$

which is equivalent to the exact dispersion relation in (9) upon making the following identification $\varepsilon^{-1} = h$. Thus, the step of the lattice ε has the interpretation of the inverse of string tension. Then, using (2), (11), we conclude that $J = \sqrt{\lambda}\mathcal{J}$, which is the familiar relation between the semiclassical $\mathcal{J}$ and exact J angular momentum.

Let us now turn to the question of the exact generalization of the tree-level expression for the S-matrix (3) (for the pure RR case $q = 0$, see [5]). For $q = 0$ the standard relations between the Zhukovsky variables $x^{\pm} = x^{\pm}(p)$ and the energy and momentum are as in the $AdS_5 \times S^5$ case [7]:

$$\mathrm{e}^{ip} = \frac{x^+}{x^-}, \quad e + 1 = ih(x^- - x^+), \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{h}. \tag{14}$$

To generalize to the $q \neq 0$ case, let us first modify the relations between the Zhukovsky variables $x^{\pm}$ and e, p (14) as follows:

$$\mathrm{e}^{ip} = \frac{x^+_{\pm}}{x^-_{\pm}}, \quad e_{\pm} + 1 \pm 2hq \sin\frac{p}{2} = ih\hat{q}(x^-_{\pm} - x^+_{\pm}). \tag{15}$$

In terms of $x^{\pm}_{\pm}$ the dispersion relation (9) takes the following form:

$$\hat{q}\left(x^+_{\pm} + \frac{1}{x^+_{\pm}} - x^-_{\pm} - \frac{1}{x^-_{\pm}}\right) \pm 2q\left(\sqrt{\frac{x^-_{\pm}}{x^+_{\pm}}} - \sqrt{\frac{x^+_{\pm}}{x^-_{\pm}}}\right) = \frac{2i}{h}. \tag{16}$$

Solving for $x^{\pm}_{\pm}$ in terms of $e_{\pm}$ and p for general q, we find

$$x^{\pm}_{\pm} = r_{\pm}\,\mathrm{e}^{\pm\frac{ip}{2}}, \quad r_{\pm} = \frac{e_{\pm} + 1 \pm 2hq \sin p/2}{2h\hat{q} \sin p/2} = \frac{2h\hat{q} \sin p/2}{e_{\pm} - 1 \mp 2hq \sin p/2}. \tag{17}$$

As this rescaling by $\hat{q}$ does not affect the action of the supersymmetry algebra, we can write down the functions parameterizing the exact S-matrix (3) in the case of $q \neq 0$,

$$\begin{aligned}
L_{1\pm} &= \mathrm{S}_{1\pm}, & \Lambda_{1\pm} &= \mathrm{S}_{1\pm}\sqrt{\frac{x^-_{\pm}x'^+_{\pm}}{x^+_{\pm}x'^-_{\pm}}}\,\frac{x^+_{\pm} - x'^-_{\pm}}{x^-_{\pm} - x'^+_{\pm}},\\
L_{3\pm} &= \mathrm{S}_{1\pm}\sqrt{\frac{x^-_{\pm}}{x^+_{\pm}}}\,\frac{x^+_{\pm} - x'^+_{\pm}}{x^-_{\pm} - x'^+_{\pm}}, & \Lambda_{3\pm} &= \mathrm{S}_{1\pm}\sqrt{\frac{x'^+_{\pm}}{x'^-_{\pm}}}\,\frac{x^-_{\pm} - x'^-_{\pm}}{x^-_{\pm} - x'^+_{\pm}},\\
L_{5\pm} &= -i\,\frac{\alpha_{\pm}}{\alpha'_{\pm}}\,\mathrm{S}_{1\pm}\sqrt{\frac{x^-_{\pm}x'^+_{\pm}}{x^+_{\pm}x'^-_{\pm}}}\,\frac{\eta_{\pm}\eta'_{\pm}}{x^-_{\pm} - x'^+_{\pm}}, & \Lambda_{5\pm} &= -i\,\frac{\alpha'_{\pm}}{\alpha_{\pm}}\,\mathrm{S}_{1\pm}\,\frac{\eta_{\pm}\eta'_{\pm}}{x^-_{\pm} - x'^+_{\pm}},\\
L_{6\pm} &= \mathrm{S}_{2\pm}, & \Lambda_{6\pm} &= \mathrm{S}_{2\pm}\sqrt{\frac{x^-_{\pm}x'^-_{\mp}}{x^+_{\pm}x'^+_{\mp}}}\,\frac{1 - x^+_{\pm}x'^+_{\mp}}{1 - x^-_{\pm}x'^-_{\mp}},\\
L_{2\pm} &= \mathrm{S}_{2\pm}\sqrt{\frac{x^-_{\pm}}{x^+_{\pm}}}\,\frac{1 - x^+_{\pm}x'^-_{\mp}}{1 - x^-_{\pm}x'^-_{\mp}}, & \Lambda_{2\pm} &= \mathrm{S}_{2\pm}\sqrt{\frac{x'^-_{\mp}}{x'^+_{\mp}}}\,\frac{1 - x^-_{\pm}x'^+_{\mp}}{1 - x^-_{\pm}x'^-_{\mp}},\\
L_{4\pm} &= i\alpha_{\pm}\alpha'_{\mp}\,\mathrm{S}_{2\pm}\sqrt{\frac{x^-_{\pm}x'^-_{\mp}}{x^+_{\pm}x'^+_{\mp}}}\,\frac{\eta_{\pm}\eta'_{\mp}}{1 - x^-_{\pm}x'^-_{\mp}}, & \Lambda_{4\pm} &= i\,\frac{1}{\alpha_{\pm}\alpha'_{\mp}}\,\mathrm{S}_{2\pm}\,\frac{\eta_{\pm}\eta'_{\mp}}{1 - x^-_{\pm}x'^-_{\mp}}.
\end{aligned} \tag{18}$$

As the S-matrix at $q = 0$ satisfies the Yang–Baxter equation without the need to use the dispersion relation, and since it has a block-diagonal structure, the above generalization of this S-matrix to the $q \neq 0$ case should still satisfy the YBE.

Let us recall that the tree-level S-matrix's generalization to non-zero q [1] was remarkably simple. In particular, the functions $l_{1,2,3}$ only depend on q through the dispersion relation. This simplicity is apparent in the exact S-matrix when written in terms of the Zhukovsky variables. It is worth noting that if the exact S-matrix is written in terms of the energy and momentum variables, this simplicity is no longer manifest due to the nontrivial all-order definition of the Zhukovsky variables (17) (in particular, their dependence on q is not only through the dispersion relation). This is hinted at in the tree-level results by the more complicated structure of $l_{4,5}$ compared to $l_{1,2,3}$ (4).

As for the four phases $\mathrm{S}_{1\pm}$, $\mathrm{S}_{2\pm}$, like their $q = 0$ limits [5], they are not fixed by the symmetry or the Yang–Baxter equation. Observing that $(x^{\pm}_{\pm})^* = x^{\mp}_{\pm}$, it can be seen that the S-matrix in the $q \neq 0$ case is QFT-unitary, while for braiding unitarity the phases should satisfy additional constraints

$$\begin{aligned} &\mathrm{S}_{1\pm}(x^+_\pm, x^-_\pm; x'^+_\pm, x'^-_\pm)\,\mathrm{S}_{1\pm}(x'^+_\pm, x'^-_\pm; x^+_\pm, x^-_\pm) = 1, \\ &\mathrm{S}_{2\pm}(x^+_\pm, x^-_\pm; x'^+_\mp, x'^-_\mp)\,\mathrm{S}_{2\mp}(x'^+_\mp, x'^-_\mp; x^+_\pm, x^-_\pm) = \sqrt{\frac{x^+_\pm x'^+_\mp}{x^-_\pm x'^-_\mp}\,\frac{1 - x^-_\pm x'^-_\mp}{1 - x^+_\pm x'^+_\mp}}. \end{aligned} \tag{19}$$

It is then natural to conjecture that the pattern of the generalization to the $q \neq 0$ case described above may also apply to the phases, i.e., to find their expressions in terms of the new Zhukovsky variables, we just need to replace $x^\pm \to x^\pm_\pm$ and $x'^\pm \to x'^\pm_\pm$ in the $q = 0$ phases as

$$\mathrm{S}_{1\pm} \stackrel{?}{=} \mathrm{S}_1(x^+_\pm, x^-_\pm; x'^+_\pm, x'^-_\pm), \quad \mathrm{S}_{2\pm} \stackrel{?}{=} \mathrm{S}_2(x^+_\pm, x^-_\pm; x'^+_\mp, x'^-_\mp). \tag{20}$$

However, this prescription is ambiguous: since the dispersion relation is modified for $q \neq 0$, starting with two expressions equal at $q = 0$, using in one of them the $q = 0$ dispersion relation and then generalizing to $q \neq 0$ as in (20), we would find different results. This suggests that the expressions for the four undetermined phases should be given by some modification of (20) that resolves this ambiguity.

The exact dispersion relation and the S-matrix presented above is a starting point for the construction of the corresponding Bethe ansatz for the string spectrum in the general $q \neq 0$ case. The full S-matrix is a product [1, 5] of two copies of the "elementary" S-matrices (3) with the coefficient functions given in (18). Remarkably, these are exactly the same as in the $q = 0$ case [5], just with $_\pm$ subscripts added. The details are then encoded in the generalization of the dispersion relation to the $q \neq 0$ case according to (9) and (16). This suggests that the corresponding Bethe ansatz that corresponds to this scattering matrix should have essentially the same structure as found in the $q = 0$ case in [5]. Once again, this is largely due to the symmetry algebra being the same for any value of q. The same should apply to the construction of the corresponding Y-system and TBA equations. One outstanding open problem (already for $q = 0$) is to find the exact expressions for the four dynamical phases $\mathrm{S}_{1\pm}, \mathrm{S}_{2\pm}$ in the S-matrix (18).

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