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OFF-SHELL ACTIONS FOR CONFORMAL SUPERGRAVITY IN THREE DIMENSIONS

D. Butter [a, 1], *S. M. Kuzenko* [b, 2], *J. Novak* [b, 3], *G. Tartaglino-Mazzucchelli* [b, 4]

[a] NIKHEF Theory Group, Amsterdam

[b] School of Physics M013, The University of Western Australia, Crawley, Australia

We review the recent construction of the $\mathcal{N} \leqslant 6$ off-shell conformal supergravity actions in three dimensions. The approach makes use of a novel superspace formulation for $\mathcal{N}$-extended conformal supergravity and the superform approach to engineer supersymmetric invariants.

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INTRODUCTION

In a series of papers published between 1985 and 1989, the actions for $\mathcal{N}$-extended conformal supergravity theories in three dimensions (3D) were constructed. The $\mathcal{N} = 1$ action was given in [1]; the $\mathcal{N} = 2$ case was presented in [2]; the $\mathcal{N}$-extended case was then worked out in [3] as a supersymmetric Chern–Simons theory for the superconformal algebra $\mathfrak{osp}(\mathcal{N}|4, \mathbb{R})$. None of these actions contained auxiliary fields. The $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories turned out to be off-shell since the corresponding Weyl multiplets do not require auxiliary fields [1, 2]. For $\mathcal{N} > 3$, the actions given in [3] were on-shell.

The problem of constructing the off-shell conformal supergravity actions has only recently been solved for the cases $\mathcal{N} = 3, 4, 5$ [4] and soon after for $\mathcal{N} = 6$ [5, 6]. This report is a review of the main aspects of [4, 6].

Superspace techniques have been used to construct off-shell formulations for $\mathcal{N}$-extended supergravity in 3D. A comprehensive analysis of the $\mathcal{N} = 1$ case was given in [7]. The $\mathcal{N} \geqslant 2$ cases were sketched in [8] and then recently developed in [9]. The superspace formulation of [8, 9], which we refer to as $SO(\mathcal{N})$ superspace, is based on gauging the structure group $SL(2, \mathbb{R}) \times SO(\mathcal{N})$. Although a formalism to construct general supergravity-matter systems with $\mathcal{N} \leqslant 4$ was given in [9, 10], no superspace construction of conformal supergravity actions was considered there.

[1] E-mail: dbutter@nikhef.nl

[2] E-mail: sergei.kuzenko@uwa.edu.au

[3] E-mail: joseph.novak@uwa.edu.au

[4] E-mail: gabriele.tartaglino-mazzucchelli@uwa.edu.au

In the $\mathcal{N}=1$ case, it is reasonably straightforward to construct the conformal supergravity action as a full superspace integral [7, 11–13]. The action is given by

$$S_{\rm CSG} = -\int d^3x\, d^2\theta E \Omega^{\alpha\beta\gamma}\left\{C_{\alpha\beta\gamma} - \frac{4}{3}\varepsilon_{\alpha(\beta}\mathcal{D}_{\gamma)}\mathcal{S}\right\} + 16i\int d^3x\, d^2\theta E\mathcal{S}^2 + \\ + \frac{1}{3}\int d^3x\, d^2\theta E\left\{\frac{1}{4}\Omega^{\alpha}{}_{\gamma}{}^{\delta}\Omega^{\beta}{}_{\delta}{}^{\rho}\Omega_{\alpha\beta\rho}{}^{\gamma} - 2\mathcal{S}\Omega^{\alpha}{}_{\beta}{}^{\gamma}\Omega_{\alpha\gamma}{}^{\beta} - \mathcal{S}\Omega^{\alpha\beta}{}_{\alpha}\Omega^{\gamma}{}_{\beta\gamma}\right\}. \quad (1)$$

The result is built out of the supervielbein $E_A{}^M$, with $E^{-1} = \mathrm{Ber}\,(E_A{}^M)$, the Lorentz connection $\Omega_A{}^{\beta\gamma}$, and the curvature superfields $\mathcal{S}$ and $C_{\alpha\beta\gamma} = C_{(\alpha\beta\gamma)}$ that parameterize the superspace covariant derivative algebra. However, an analogous construction is impossible for $\mathcal{N} \geqslant 2$. Nevertheless, the conformal supergravity actions with $\mathcal{N} \leqslant 6$ were constructed in [4,6] by using: (i) a novel formulation of $\mathcal{N}$-extended conformal supergravity, the so-called *conformal superspace* [14]; (ii) the superform techniques [15, 16] to construct supersymmetric invariants.

1. CLOSED SUPERFORMS AND SUPERSYMMETRIC INVARIANTS

In this section, we present some generalities about the superform formalism to construct supersymmetric invariants [15, 16].

We consider a curved superspace $\mathcal{M}^{d|f}$ parameterized by local coordinates $z^M = (x^m, \theta^{\underline{\mu}})$, where d and f are, respectively, the real dimensions of the bosonic (x) and fermionic (θ) coordinates. The curved superspace is endowed with the covariant derivatives

$$\nabla_A := E_A - \omega_A, \quad E_A := E_A{}^M(z)\partial_M, \quad \omega_A := \omega_A{}^{\underline{b}}(z)X_{\underline{b}}, \quad (2)$$

where $\omega_A{}^{\underline{b}}$ is the connection associated with the local structure group, $\mathcal{H}$, with generators $X_{\underline{b}}$. The vielbein and connection one-forms are defined by

$$E^A := dz^M E_M{}^A, \quad \omega^{\underline{a}} := dz^M \omega_M{}^{\underline{a}} = E^A\omega_A{}^{\underline{a}}. \quad (3)$$

The covariant derivatives satisfy the algebra

$$[\nabla_A, \nabla_B\} = -T_{AB}{}^C\nabla_C - R_{AB}{}^{\underline{c}}X_{\underline{c}}, \quad (4)$$

where $T_{AB}{}^C$ and $R_{AB}{}^{\underline{c}}$ are, respectively, the torsion and curvatures that are associated with the two-forms $T^A := (1/2)E^C \wedge E^B T_{BC}{}^A$ and $R^{\underline{a}} := (1/2)E^C \wedge E^B R_{BC}{}^{\underline{a}}$. The covariant derivatives transform under general coordinate and structure group transformations, which together generate the local gauge group $\mathcal{G}$. The transformations of the covariant derivatives are

$$\delta_{\mathcal{G}}\nabla_A = [\mathcal{K}, \nabla_A], \quad \mathcal{K} = \xi^C(z)\nabla_C + \Lambda^{\underline{a}}(z)X_{\underline{a}}. \quad (5)$$

The spinor index $\underline{\mu}$, the group $\mathcal{H}$, the action of $X_{\underline{a}}$ on the covariant derivatives, the torsion constraints, and other important details depend on the specific curved superspace under consideration. For our purposes we leave these details unspecified.

Let us now construct supersymmetric invariants via superforms. Consider a closed d-form

$$\mathfrak{J} = \frac{1}{d!}E^{A_d}\wedge\ldots\wedge E^{A_1}\mathfrak{J}_{A_1\cdots A_d} = \frac{1}{d!}dz^{M_d}\wedge\ldots\wedge dz^{M_1}\mathfrak{J}_{M_1\cdots M_d}, \quad d\mathfrak{J} = 0. \quad (6)$$

Given such a superform, we introduce the action ($\varepsilon^{m_1\cdots m_d} := \varepsilon^{a_1\cdots a_d} e_{a_1}{}^{m_1}\cdots e_{a_d}{}^{m_d}$)

$$S = \int_{\mathcal{M}^d} \mathfrak{J} = \int d^d x\, e\, {}^*\mathfrak{J}|, \quad {}^*\mathfrak{J} = \frac{1}{d!}\varepsilon^{m_1\cdots m_d}\mathfrak{J}_{m_1\cdots m_d}, \quad e = \det\left(e_m{}^a\right), \tag{7}$$

where the bar-projection of a superfield $V(z) = V(x,\theta)$ is defined by $V| := V(x,\theta)|_{\theta=0}$. It is simple to observe that under an infinitesimal general coordinate transformation, generated by a vector field $\xi = \xi^A E_A = \xi^M \partial_M$, the d-form $\mathfrak{J}$ varies as

$$\delta_\xi \mathfrak{J} = \mathcal{L}_\xi \mathfrak{J} \equiv i_\xi d\mathfrak{J} + d i_\xi \mathfrak{J} = d i_\xi \mathfrak{J}. \tag{8}$$

Since the variation $\delta_\xi \mathfrak{J}$ is an exact d-form, the action S is then invariant up to boundary terms (that we neglect).

Suitable actions must also be invariant under the structure group transformations and, if relevant, under other symmetry transformations[1]. If the closed d-form $\mathfrak{J}$ also transforms by an exact form under the $\mathcal{H}$-transformations,

$$\delta_{\mathcal{H}} \mathfrak{J} = d\Theta(\Lambda^{\underline{a}}), \tag{9}$$

then the functional (7) is invariant under the full local gauge group $\mathcal{G}$, Eq. (5).

It is possible to formulate a prescription to construct the conformal supergravity action by using superforms in $SO(\mathcal{N})$ superspace [13].

The $SO(\mathcal{N})$ superspace of [8, 9] is based on the supermanifold $\mathcal{M}^{3|2\mathcal{N}}$ locally parameterized by $z^M = (x^m, \theta^\mu_I)$, where $m = 0,1,2$, $\mu = 1,2$ and $I = 1,\ldots,\mathcal{N}$. Its structure group is chosen to be $\mathcal{H} = SL(2,\mathbb{R})\times SO(\mathcal{N})$ with Lorentz and $SO(\mathcal{N})$ generators $X_{\underline{a}} = \{M_{ab}, N_{IJ}\}$. The $SO(\mathcal{N})$ superspace describes conformal supergravity because the geometry is invariant under super-Weyl transformations [9].

To construct the conformal supergravity action, [13] put forward the idea of using an appropriate closed three-form $\mathfrak{J}$ by making use of a two-parameter deformation of the vector covariant derivative

$$\nabla_{\alpha\beta} \to \mathfrak{D}_{\alpha\beta} = \nabla_{\alpha\beta} + \lambda \mathcal{S} M_{\alpha\beta} + \rho C_{\alpha\beta}{}^{KL} N_{KL}, \tag{10}$$

where λ and ρ are real parameters, and $\mathcal{S}$ and $C_{\alpha\beta}{}^{KL} = C_{\alpha\beta}{}^{[KL]}$ are two curvature superfields. The deformed covariant derivatives $\mathfrak{D}_A = (\mathfrak{D}_a, \mathfrak{D}^I_\alpha) := (\mathfrak{D}_a, \nabla^I_\alpha)$ obey the algebra

$$[\mathfrak{D}_A, \mathfrak{D}_B\} = -\mathbf{T}_{AB}{}^C \mathfrak{D}_C - \frac{1}{2}\mathbf{R}_{AB}{}^{cd} M_{cd} - \frac{1}{2}\mathbf{R}_{AB}{}^{KL} N_{KL}. \tag{11}$$

Using the new curvatures, one considers the superform equation

$$d\mathbf{\Sigma} = \frac{1}{2}\mathbf{R}^{ab}\wedge \mathbf{R}_{ab} + \frac{\kappa}{2}\mathbf{R}^{IJ}\wedge \mathbf{R}_{IJ}, \tag{12}$$

with κ being a real parameter, and looks for two solutions $\mathbf{\Sigma}_{\rm CS}$ and $\mathbf{\Sigma}_T$. The first is a Chern–Simons form, while the second is directly built out of the curvature superfields.

[1] An important example is given by the super-Weyl transformations in the context of conformal supergravity formulated in $SO(\mathcal{N})$ superspace, see [4, 9, 13].

The three-forms $\mathbf{\Sigma}_{\rm CS}$ and $\mathbf{\Sigma}_T$ can be used to construct a two-parameter family of closed forms via their difference $\mathbf{\Sigma}_{\rm CS} - \mathbf{\Sigma}_T$. Next, it is necessary to determine which linear combination $\mathfrak{J}$ of these is *super-Weyl invariant* modulo exact contribution. The parameter κ is expected to be fixed by this requirement. It is also expected that $\mathfrak{J}$ is independent of λ and ρ, due to its uniqueness. This method has been used in [13] to construct the $\mathcal{N} = 1$ case. However, the method is involved, and not well adapted to the construction of conformal supergravity actions. The problem simplifies if one uses the 3D *conformal superspace* of [14]. This was indeed the main reason why we developed such a formalism.

2. WEYL MULTIPLET IN 3D CONFORMAL SUPERSPACE

The 3D $\mathcal{N}$-extended conformal superspace is also based on the supermanifold $\mathcal{M}^{3|2\mathcal{N}}$. Compared to $SO(\mathcal{N})$ superspace, the structure group $\mathcal{H}$ is enlarged to include the generators $X_{\underline{a}} = \{M_{ab}, N_{IJ}, \mathbb{D}, K_A\}$, where $\mathbb{D}$ is the dilatation generator, $K_A = (K_a, S^I_\alpha)$ are the special (super)conformal generators with conformal boosts (K_a) and S-supersymmetry (S^I_α). Together with the super-Poincaré translations $P_A = (P_a, Q^I_\alpha)$, the operators $X_{\tilde{a}} = \{P_A, X_{\underline{a}}\}$ describe the generators of the 3D $\mathcal{N}$-extended superconformal algebra, $\mathfrak{osp}(\mathcal{N}|4,\mathbb{R})$. Their algebra is given explicitly in [14]. The covariant derivatives have the form

$$\nabla_A = E_A - \omega_A{}^{\underline{b}} X_{\underline{b}} = E_A - \frac{1}{2}\Omega_A{}^{bc} M_{bc} - \frac{1}{2}\Phi_A{}^{JK} N_{JK} - B_A \mathbb{D} - \mathfrak{F}_A{}^B K_B. \tag{13}$$

The entire superconformal group $\mathfrak{osp}(\mathcal{N}|4,\mathbb{R})$ is gauged in superspace as a result of the supergravity gauge transformations $\mathcal{G}$, Eq. (5), see [14] for details. It is important to note that the action of the generators $X_{\underline{a}}$ on the covariant derivatives

$$[X_{\underline{a}}, \nabla_B\} = -f_{\underline{a}B}{}^C \nabla_C - f_{\underline{a}B}{}^{\underline{c}} X_{\underline{c}} \tag{14}$$

resembles that with P_A in the superconformal algebra

$$[X_{\underline{a}}, P_B\} = -f_{\underline{a}B}{}^C P_C - f_{\underline{a}B}{}^{\underline{c}} X_{\underline{c}}. \tag{15}$$

To describe the Weyl multiplet, the torsion and curvatures of conformal superspace have to be constrained. The constraints chosen in [14] were such that: (i) the entire covariant derivative algebra is expressed in terms of a single primary superfield, the $\mathcal{N}$-extended super-Cotton tensor; and (ii) the superspace geometry resembles the one describing the Yang–Mills supermultiplet. The anticommutator of two spinor covariant derivatives with $\mathcal{N} > 3$ is

$$\begin{aligned}\{\nabla^I_\alpha, \nabla^J_\beta\} = 2i\delta^{IJ}\nabla_{\alpha\beta} + i\varepsilon_{\alpha\beta}W^{IJKL}N_{KL} - \frac{i}{\mathcal{N}-3}\varepsilon_{\alpha\beta}(\nabla^\gamma_K W^{IJKL})S_{\gamma L} + \\ + \frac{1}{2(\mathcal{N}-2)(\mathcal{N}-3)}\varepsilon_{\alpha\beta}(\gamma^c)^{\gamma\delta}(\nabla_{\gamma K}\nabla_{\delta L}W^{IJKL})K_c.\end{aligned} \tag{16}$$

The antisymmetric superfield $W^{IJKL} = W^{[IJKL]}$ is the super-Cotton tensor for $\mathcal{N} > 3$. It is a conformal primary $(S^I_\alpha W^{JKLP} = 0)$ of dimension-1 $(\mathbb{D}W^{IJKL} = W^{IJKL})$ and satisfies the Bianchi identity[1]

$$\nabla^I_\alpha W^{JKLP} = \nabla^{[I}_\alpha W^{JKLP]} - \frac{4}{\mathcal{N}-3}\nabla_{\alpha Q} W^{Q[JKL}\delta^{P]I}. \tag{17}$$

With $\mathcal{N} \leqslant 3$, W^{IJKL} is zero. The properties of the super-Cotton tensor and the algebra of covariant derivatives vary for each $\mathcal{N} \leqslant 3$ case [14]. Here we review only the $\mathcal{N} > 3$ cases.

The 3D $\mathcal{N}$-extended Weyl multiplet involves a set of gauge fields: the vielbein $e_m{}^a$, the gravitino $\psi_m{}^\alpha_I$, the $SO(\mathcal{N})$ gauge field $V_m{}^{IJ}$, and the dilatation gauge field b_m. They appear as the lowest components of the vielbein, SO($\mathcal{N}$) and dilatation connection

$$e_m{}^a := E_m{}^a|, \quad \psi_m{}^\alpha_I := 2E_m{}^\alpha_I|, \quad V_m{}^{IJ} := \Phi_m{}^{IJ}|, \quad b_m := B_m|. \tag{18}$$

The other connections are defined in terms of the previous ones [4]. The nontrivial components of the super-Cotton tensor comprise the remaining physical fields of the off-shell Weyl multiplet and a set of composite fields [4]. The unconstrained auxiliary fields are

$$w_{IJKL} := W_{IJKL}|\,, \qquad w_\alpha{}^{IJK} := -\frac{i}{2(\mathcal{N}-3)}\nabla_{\alpha L} W^{IJKL}\Big|, \tag{19a}$$

$$X_\alpha{}^{I_1\cdots I_5} := i\nabla^{[I_1}_\alpha W^{I_2\cdots I_5]}\Big|, \qquad y^{IJKL} := \frac{i}{\mathcal{N}-3}\nabla^{\gamma[I}\nabla_{\gamma P} W^{JKL]P}\Big|. \tag{19b}$$

For $\mathcal{N} > 5$, the super-Cotton tensor includes another set of physical fields defined by

$$X_{\alpha_1\cdots\alpha_n}{}^{I_1\cdots I_{n+4}} := I(n)\nabla^{[I_1}_{(\alpha_1}\cdots\nabla^{I_n}_{\alpha_n)} W^{I_{n+1}\cdots I_{n+4}]}\Big|, \tag{20}$$

where $I(n) = i$ for $n = 1, 2 \pmod 4$ and $I(n) = 1$ for $n = 3, 4 \pmod 4$. These fields are the field strengths of hidden (super)symmetries of the Weyl multiplet. The simplest example is given by $\mathcal{N} = 6$. In this case we have (up to contributions involving the gravitini)

$$X_{\alpha\beta}{}^{I_1\cdots I_6} = -\frac{1}{2}\varepsilon^{I_1\cdots I_6}(\gamma_a)_{\alpha\beta}\varepsilon^{abc}\mathcal{F}_{bc} + \mathcal{O}(\psi), \tag{21}$$

where $\mathcal{F}_{ab} = 2e_{[a}{}^m e_{b]}{}^n \partial_m A_n$ is the field strength for an extra $U(1)$ gauge field A_b. This property has a crucial role in constructing the conformal supergravity action for $\mathcal{N} = 6$.

3. CONFORMAL SUPERGRAVITY ACTION

In the superform construction of the off-shell conformal supergravity action in conformal superspace, a natural ingredient is the $\mathfrak{osp}(\mathcal{N}|4,\mathbb{R})$ Chern–Simons superform $\Sigma_{\rm CS}$. Given the structure constants $f_{\tilde a\tilde b}{}^{\tilde c}$ and Killing metric $\Gamma_{\tilde a\tilde b} = (-1)^{\varepsilon_{\tilde c}} f_{\tilde a\tilde d}{}^{\tilde c} f_{\tilde b\tilde c}{}^{\tilde d}$ of $\mathfrak{osp}(\mathcal{N}|4,\mathbb{R})$, see [4] for their definition and normalization, we find $\Sigma_{\rm CS}$ to be

$$\Sigma_{\rm CS} = R^{\tilde b}\wedge\omega^{\tilde a}\Gamma_{\tilde a\tilde b} + \frac{1}{6}\omega^{\tilde c}\wedge\omega^{\tilde b}\wedge\omega^{\tilde a} f_{\tilde a\tilde b}{}^{\tilde d}\Gamma_{\tilde d\tilde c}. \tag{22}$$

[1] In the $\mathcal{N} = 4$ case, $W^{IJKL} = \varepsilon^{IJKL}W$ and Eq. (17) is identically satisfied. In this case, the Bianchi identity for W is $\nabla^{\alpha I}\nabla^J_\alpha W = (1/4)\delta^{IJ}\nabla^\alpha_P\nabla^P_\alpha W$.

Note that in conformal superspace, we may treat the vielbein on the same footing as the connections $\omega^{\tilde{a}} := (E^A, \omega^{\underline{a}})$. We define the curvature two-form of conformal superspace as $R^{\tilde{a}} := (R(P)^A, R^{\underline{a}}) = (1/2)E^B \wedge E^A R_{AB}{}^{\tilde{a}}$, where $R(P)^A := T^A - \hat{T}^A$, and $\hat{T}^A$ is the flat superspace torsion [4]. The CS form varies as an exact form under structure group transformations

$$\delta_{\mathcal{H}}\Sigma_{\rm CS} = d(d\omega^{\tilde{b}}\Lambda^{\underline{a}}\Gamma_{\underline{a}\tilde{b}}) \tag{23}$$

and satisfies the superform equation

$$d\Sigma_{\rm CS} = \langle R^2\rangle, \quad \langle R^2\rangle := R^{\tilde{b}} \wedge R^{\tilde{a}}\Gamma_{\tilde{a}\tilde{b}}. \tag{24}$$

The explicit expression for $\Sigma_{\rm CS}$ in terms of various curvatures and connections is

$$\Sigma_{\rm CS} = -\hat{R}^a \wedge \Omega_a - \frac{1}{6}\Omega^c \wedge \Omega^b \wedge \Omega^a \varepsilon_{abc} - 4iE^a \wedge \mathfrak{F}^{\alpha I} \wedge \mathfrak{F}_I^{\beta}(\gamma_a)_{\alpha\beta} - \hat{R}^{IJ} \wedge \Phi_{IJ} + \\ + \frac{1}{3}\Phi^{IJ} \wedge \Phi_I{}^K \wedge \Phi_{KJ} + 2E^a \wedge \mathfrak{F}_a \wedge B - 2E_I^{\alpha} \wedge \mathfrak{F}_{\alpha}^I \wedge B + \text{exact form}, \tag{25}$$

where $\hat{R}^{ab} := d\Omega^{ab} + \Omega^{ac} \wedge \Omega_c{}^b$ and $\hat{R}^{IJ} := d\Phi^{IJ} + \Phi^{IK} \wedge \Phi_K{}^J$ correspond to the Riemann and SO($\mathcal{N}$) curvature tensors, respectively. The explicit form of $\langle R^2\rangle$ turns out to be

$$\langle R^2\rangle = -R(N)^{IJ} \wedge R(N)_{IJ}, \tag{26}$$

with $R(N)^{IJ} = (1/2)E^B \wedge E^A R_{AB}{}^{IJ}(N)$ being the $SO(\mathcal{N})$ curvature two-form of conformal superspace. In the $\mathcal{N} = 1, 2$ cases, it is identically zero, $\langle R^2\rangle \equiv 0$. This implies that the action constructed by using (7) with $\mathfrak{J} = \Sigma_{\rm CS}$ is invariant under all the supergravity gauge transformations. This is consistent with the fact that the $\mathcal{N} = 1, 2$ results of [1,2] are off-shell. On the other hand, for $\mathcal{N} > 2$, $\langle R^2\rangle \neq 0$ and Eq. (24) implies that the CS form alone cannot be used to define a locally supersymmetric action principle. However, this was resolved in [4]. The idea in [4], inspired by [13] and the 4D construction of [17], was to search for the second solution to the superform equation

$$d\Sigma = \langle R^2\rangle, \quad 4\nabla_{[A}\Sigma_{BCD)} + 6T_{[AB}{}^E\Sigma_{|E|CD)} = \langle R^2\rangle_{ABCD}. \tag{27}$$

Since we expect the CS form to generate the on-shell conformal supergravity action, we search for a second superform defined in terms of the auxiliary fields of the Weyl multiplet. The other solution Σ_R, which we refer to as the curvature induced form, should then be constructed only in terms of the super-Cotton tensor and its covariant derivatives. It turns out to be invariant $\delta_{\mathcal{H}}\Sigma_R = 0$. If Σ_R exists, $\mathfrak{J} = \Sigma_{\rm CS} - \Sigma_R$ is an appropriate closed form describing the action of conformal supergravity.

Let us now search for Σ_R with $\mathcal{N} > 3$. Since we want it to be constructed in terms of W^{IJKL}, the only possible ansatz for the lowest components of Σ_R is [4]

$$\Sigma_{\alpha\beta\gamma}^{I\,JK} = 0, \quad \Sigma_{a\,\beta\gamma}^{\;JK} = i\,(\gamma_a)_{\beta\gamma}\left(\mathcal{A}\,\delta^{JK}W^{ILPQ}W_{ILPQ} + \mathcal{B}W^{LPQJ}W_{LPQ}{}^K\right), \tag{28}$$

with $\mathcal{A}$ and $\mathcal{B}$ being two constants to be determined. Plugging the previous ansatz into Eq. (27), gives us at the lowest dimension (the wedge products are suppressed)

$$0 = E_L^{\alpha}E_{\alpha K}E_J^{\beta}E_{\beta I}\Big[W^{PQIJ}W^{KL}{}_{PQ} - \\ - \left(\mathcal{A}W^{PQRS}W_{PQRS}\delta^{J[K} + \mathcal{B}W^{PQRJ}W_{PQR}{}^{[K}\right)\delta^{L]I}\Big]. \tag{29}$$

The first term in the previous expression contains a double traceless contribution of the form

$$\left(\delta^R_{[K}\delta^{[I}_{|S|} - \frac{1}{\mathcal{N}}\delta^R_S\delta^{[I}_{[K}\right)\left(\delta^{|T|}_{L]}\delta^{J]}_U - \frac{1}{\mathcal{N}}\delta^{J]}_{L]}\delta^T_U\right) W^{SUPQ}W_{RTPQ}, \tag{30}$$

which cannot be cancelled for $\mathcal{N} > 5$. On the other hand, for $\mathcal{N} = 4, 5$, Eq. (29) can be solved and the three-form Σ_R constructed. Also for $\mathcal{N} = 3$, Σ_R exists, although its structure is different since the super-Cotton tensor in this case is a spinor superfield W_α [4].

Even though the approach of [4] succeeded in constructing for the first time the $\mathcal{N} = 3, 4, 5$ conformal supergravity actions, it remained unclear why the construction did not work for $\mathcal{N} > 5$. The answer was found in [6] for the $\mathcal{N} = 6$ case. We now focus on this case.

There is a simple reason why the construction failed: the hidden $U(1)$ symmetry associated with the component $X_{\alpha\beta}{}^{I_1\cdots I_6}$ of the $\mathcal{N} = 6$ Weyl multiplet was not taken into account. It turns out that, in the $\mathcal{N} = 6$ case, the Hodge dual of the Cotton tensor $W^{IJ} := (1/4!)\varepsilon^{IJKLPQ}W_{KLPQ}$ satisfies the Bianchi identity for the field strength of an $\mathcal{N} = 6$ Abelian vector multiplet

$$\nabla^{[I}_\alpha W^{JK]} = \nabla^{[I}_\alpha W^{JK]} - \frac{2}{5}\delta^{I[J}\nabla_{\alpha L}W^{K]L}. \tag{31}$$

Therefore, by using W^{IJ} one can define a closed two-form $F = (1/2)E^B \wedge E^A F_{AB}$, $dF = 0$,

$$\begin{aligned} F^{IJ}_{\alpha\beta} &= -2i\varepsilon_{\alpha\beta}W^{IJ}, \quad F^{\ I}_{a\alpha} = \frac{1}{5}(\gamma_a)_\alpha{}^\beta\nabla_{\beta J}W^{IJ}, \\ F_{ab} &= -\frac{i}{120}\varepsilon_{abc}(\gamma^c)^{\alpha\beta}[\nabla^K_\alpha, \nabla^L_\beta]W_{KL}. \end{aligned} \tag{32}$$

Associated with the field strength F to be a gauge one-form A, $F = dA$. By using A and F we now modify the $\mathfrak{osp}(\mathcal{N}|4, \mathbb{R})$ Chern–Simons form (22) by adding the $U(1)$ CS term

$$\Sigma'_{\rm CS} := \Sigma_{\rm CS} - \mathcal{C}F\wedge A, \quad d\Sigma'_{\rm CS} = \langle R^2\rangle - \mathcal{C}F\wedge F, \tag{33}$$

where $\mathcal{C}$ is some undetermined constant. With these modifications made, the curvature induced form with ansatz (28) turns out to solve the equation $d\Sigma_R = \langle R^2\rangle - \mathcal{C}F\wedge F$ provided we choose $\mathcal{C} = -2$. The nonzero components of Σ_R are found to be [6]

$$\Sigma_{a\beta\gamma}^{\ \ JK} = 8i(\gamma_a)_{\beta\gamma}\left[W^{JP}W^K{}_P - \frac{1}{4}\delta^{JK}W^{PQ}W_{PQ}\right], \tag{34a}$$

$$\Sigma_{ab\gamma}^{\ \ K} = -2\varepsilon_{abc}(\gamma^c)_\gamma{}^\delta\left[(\nabla^{[K}_\delta W^{PQ]})W_{PQ} - \frac{2}{5}(\nabla^P_\delta W_{QP})W^{QK}\right], \tag{34b}$$

$$\begin{aligned} \Sigma_{abc} = i\varepsilon_{abc}\Bigg[&\frac{1}{5}(\nabla^{\gamma I}\nabla_{\gamma K}W^{JK})W_{IJ} + \frac{1}{3}(\nabla^{\gamma[I}W^{JK]})\nabla_{\gamma[I}W_{JK]} - \\ &- \frac{2}{25}(\nabla^\gamma_I W^{KI})\nabla^J_\gamma W_{KJ} - \frac{i}{3}\varepsilon^{IJKLPQ}W_{IJ}W_{KL}W_{PQ}\Bigg]. \end{aligned} \tag{34c}$$

Now the closed three-form $\mathfrak{J} = \Sigma'_{\rm CS} - \Sigma_R$ generates a locally supersymmetric action according to the rule (7). It is a straightforward exercise to express the resulting action for $\mathcal{N} = 6$

conformal supergravity in components. The action is [6]

$$S = \frac{1}{4}\int d^3x\, e\left\{\varepsilon^{abc}\left(\omega_a{}^{fg}\mathcal{R}_{bcfg} - \frac{2}{3}\omega_{af}{}^{g}\omega_{bg}{}^{h}\omega_{ch}{}^{f} - \frac{i}{2}\Psi_{bc}{}^{\alpha}_{I}(\gamma_d)_\alpha{}^\beta(\gamma_a)_\beta{}^\gamma\varepsilon^{def}\Psi_{ef}{}^{I}_{\gamma} - \right.\right.$$
$$\left. - 2\mathcal{R}_{ab}{}^{IJ}V_{cIJ} - \frac{4}{3}V_a{}^{IJ}V_{bI}{}^{K}V_{cKJ} + 4\mathcal{F}_{ab}A_c\right) 4y^{IJ}w_{IJ} - \frac{16i}{3}\tilde{w}^{\alpha IJK}\tilde{w}_{\alpha IJK} +$$
$$+ 8iX_K^\gamma X_\gamma^K - \frac{8}{3}\varepsilon^{IJKLPQ}w_{IJ}w_{KL}w_{PQ} - 8i\psi_a{}^{\alpha}_{I}(\gamma^a)_\alpha{}^\beta(\tilde{w}_\beta{}^{IJK}w_{JK} + X_{\beta J}w^{IJ}) +$$
$$\left. + 4i\varepsilon^{abc}(\gamma_a)_{\alpha\beta}\psi_b{}^{\alpha}_{I}\psi_c{}^{\beta}_{J}\left(w^{IK}w^J{}_K - \frac{1}{4}\delta^{IJ}w^{KL}w_{KL}\right)\right\}, \quad (35)$$

where $\mathcal{R}_{ab}{}^{cd}$, $\Psi_{bc}{}^{\alpha}_{I}$, $\mathcal{R}_{ab}{}^{IJ}$, and $\mathcal{F}_{ab}$ represent the component Riemann curvature, gravitini field strength, $SO(6)$ curvature, and $U(1)$ field strength, respectively. We have also used for $\mathcal{N}=6$ the following definitions of the Weyl multiplet's auxiliary fields:

$$w^{IJ} := \frac{1}{4!}\varepsilon^{IJKLPQ}W_{KLPQ}\Big|, \quad y^{IJ} := -\frac{i}{5}\nabla^{\gamma[I}\nabla_{\gamma P}W^{J]P}\Big| - \frac{1}{2}\varepsilon^{IJKLPQ}W_{KL}W_{PQ}\Big|, \quad (36)$$

$$\tilde{w}_\alpha{}^{IJK} := -\frac{i}{2}\nabla_\alpha^{[I}W^{JK]}\Big|, \quad X_\alpha{}^I := -\frac{i}{5}\nabla_{\alpha J}W^{IJ}\Big|. \quad (37)$$

These are the $\mathcal{N}=6$ Hodge duals of the fields defined in Eq. (19).

By eliminating the auxiliary fields w^{IJ}, $\tilde{w}_\alpha{}^{IJK}$, $X_\alpha{}^I$, and y^{IJ}, one is left with the on-shell action, which we denote by $S_{\rm CSG}$. It corresponds to the first two lines of (35), which come from the component projection of the Chern–Simons contribution $\Sigma'_{\rm CS}$ in $\mathfrak{J}$. The last three lines of (35) come from Σ_R and give the contribution from the auxiliary fields to the off-shell action, as expected. Note that, due to the $U(1)$ Chern–Simons term, $S_{\rm CSG}$ differs from the on-shell action for $\mathcal{N}=6$ conformal supergravity obtained in [3].

The actions for $\mathcal{N}<6$ can simply be obtained from (35) by using the truncation procedure in [4,6]. As an example, let us show how it works in the $\mathcal{N}=6 \to \mathcal{N}=5$ case [4]. We first need to switch off gauge fields possessing an index $I=6$ together with the $U(1)$ gauge field $A_b \to 0$. Then $S_{\rm CSG}$ in (35) becomes the CS contribution to the $\mathcal{N}=5$ action. As a final step, in (35) we need to truncate the $\mathcal{N}=6$ auxiliary fields to $\mathcal{N}=5$. With $I,J,K=1,2,3,4,5$ we set to zero $w_{IJ}=\tilde{w}_\alpha{}^{IJK}=X_\alpha{}^I=y^{IJ}=0$ and take

$$w^{I6} \to w^I = \frac{1}{4!}\varepsilon^{IJKLP}w_{JKLP}, \quad \tilde{w}_\alpha{}^{IJ6} \to w_\alpha{}^{IJ} = \frac{1}{3!}\varepsilon^{IJKLP}w_{\alpha KLP}, \quad (38a)$$

$$X_\alpha^6 \to X_\alpha = \frac{1}{5!}\varepsilon_{IJKLP}X_\alpha{}^{IJKLP}, \quad y^{I6} \to y^I = \frac{1}{4!}\varepsilon^{IJKLP}y_{JKLP}. \quad (38b)$$

The $\mathcal{N}=3,4$ cases can be obtained via a similar truncation procedure [4].

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