Письма в ЭЧАЯ. 2014. Т. 11, № 7(191). С. 1449–1456

ONCE MORE ON PARASTATISTICS

V. N. Tolstoy [1]

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow

Equivalence between algebraic structures, generated by parastatistics triple relations of Green (1953) and Greenberg–Messiah (1965), and certain orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras is found explicitly. Moreover, it is shown that such superalgebras give more complex para-Fermi and para-Bose systems than ones of Green–Greenberg–Messiah.

PACS: 87.19.xe

INTRODUCTION

The usual creation and annihilation operators of identical particles, fermions $a_i^{\pm}$ ($i = 1, \ldots, m$) and bosons $b_j^{\pm}$ ($j = 1, \ldots, n$), satisfy the canonical commutation relations:

$$\{a_i^{\zeta}, a_j^{\eta}\} = \frac{1}{2}|\eta - \zeta|\delta_{ij}, \quad [b_i^{\zeta}, b_j^{\eta}] = \frac{1}{2}(\eta - \zeta)\delta_{ij}. \tag{1}$$

Here and elsewhere the Greek letters $\zeta, \eta \in \{+, -\}$, if they are upper indexes, are interpreted as $+1$ and -1 in the algebraic expressions of the type $\eta - \zeta$.

From the relations (1), follows the so-called «*symmetrization postulate*» (SP): *States of more than one identical particle must be antisymmetric (fermions) or symmetric (bosons) under permutations*.

In 1953, Green [1] proposed to refuse SP, and he introduced algebras with the triple relations:

$$[[a_i^{\zeta}, a_j^{\eta}], a_k^{\xi}] = |\xi - \eta|\delta_{jk} a_i^{\zeta} - |\xi - \zeta|\delta_{ik} a_j^{\eta} \quad \text{(parafermions)}, \tag{2}$$

$$[\{b_i^{\zeta}, b_j^{\eta}\}, b_k^{\xi}] = (\xi - \eta)\delta_{jk} b_i^{\zeta} + (\xi - \zeta)\delta_{ik} b_j^{\eta} \quad \text{(parabosons)}. \tag{3}$$

The usual fermions and bosons satisfy these relations but another solutions also exist.

In 1962, Kamefuchi and Takahashi [2] (also see [3]) have shown that the parafermionic algebra is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(2m+1) := \mathfrak{o}(2m+1, \mathbb{C})$. Later, in 1980, Ganchev and Palev [4] proved that the parabosonic algebra is isomorphic to the orthosymplectic $\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1|2n)$.

[1] E-mail: vntolstoy@yandex.ru

In 1965, Greenberg and Messiah [5] considered parasystem consisting simultaneously of parafermions and parabosons, and they defined the relative commutation rules between parafermions and parabosons. There are two types of such relations:

$$\begin{aligned} &[[a_i^\zeta, a_j^\eta], b_k^\xi] = 0, && [\{b_i^\zeta, b_j^\eta\}, a_k^\xi] = 0, \\ &[[a_i^\zeta, b_j^\eta], a_k^\xi] = -|\xi - \zeta|\delta_{ik} b_j^\eta, && \{[a_i^\zeta, b_j^\eta], b_k^\xi\} = (\xi - \eta)\delta_{jk} a_i^\zeta, \end{aligned} \tag{4}$$

$$\begin{aligned} &[[a_i^\zeta, a_j^\eta], b_k^\xi] = 0, && [\{b_i^\zeta, b_j^\eta\}, a_k^\xi] = 0, \\ &\{\{a_i^\zeta, b_j^\eta\}, a_k^\xi\} = |\xi - \zeta|\delta_{ik} b_j^\eta, && [\{a_i^\zeta, b_j^\eta\}, b_k^\xi] = (\xi - \eta)\delta_{jk} a_i^\zeta, \end{aligned} \tag{5}$$

where $i, j, k = 1, 2, \ldots, m$ for the symbols a's and $i, j, k = 1, 2, \ldots, n$ for the symbols b's. The first case (4) was called as *the relative para-Fermi set*, and the second case (5) was called as *the relative para-Boson set* [1].

In 1982, Palev [6] has shown that the case (4) with (2) and (3) is isomorphic to the orthosymplectic $\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. No any similar solution for the second case (5) was known up to now.

Here we show that the case (5) is isomorphic to the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1, 2m|2n, 0)$. Moreover, it will demonstrate that the more general mixed parasystem, which simultaneously involves the relative para-Fermi and relative para-Bose sets, contains two sorts of parafermions and one sort of parabosons, and it is isomorphic to the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(2m_1+1, 2m_2|2n, 0)$. All previous cases are particular (degenerated) variants of this general case.

The paper is organized as follows. Section 1 provides a definition and general structure of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras and also a matrix realization and a Cartan–Weyl basis of the general linear $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. In Sec. 2, we describe the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(2m_1+1, 2m_2|2n, 0)$ and show that a part of its defining triple relations in the terms of short-root vectors coincides with the relative para-Bose set.

1. SUPERALGEBRA $\mathfrak{gl}(m_1, m_2|n_1, n_2)$

At first, we remind a general definition of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra [7, 8].

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded LSA $\tilde{\mathfrak{g}}$, as a linear space, is a direct sum of four graded components

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathbf{a}=(a_1,a_2)} \tilde{\mathfrak{g}}_{\mathbf{a}} = \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)}, \tag{6}$$

[1] The names *the relative para-Fermi and para-Boson set* are directly related to the type of the Lie bracket (commutator or anticommutator) given between parafermion and paraboson elements.

with a bilinear operation $[\![\cdot,\cdot]\!]$ satisfying the identities (grading, symmetry, Jacobi):

$$\deg([\![x_{\mathbf{a}},y_{\mathbf{b}}]\!]) = \deg(x_{\mathbf{a}}) + \deg(x_{\mathbf{b}}) \;=\; \mathbf{a}+\mathbf{b} = (a_1+b_1, a_2+b_2), \tag{7}$$

$$[\![x_{\mathbf{a}},y_{\mathbf{b}}]\!] = -(-1)^{\mathbf{ab}}[\![y_{\mathbf{b}},x_{\mathbf{a}}]\!], \tag{8}$$

$$[\![x_{\mathbf{a}},[\![y_{\mathbf{b}},z]\!]]\!] = [\![[\![x_{\mathbf{a}},y_{\mathbf{b}}]\!],z]\!] + (-1)^{\mathbf{ab}}[\![y_{\mathbf{b}},[\![x_{\mathbf{a}},z]\!]]\!], \tag{9}$$

where the vector (a_1+b_1, a_2+b_2) is defined $\mod(2,2)$ and $\mathbf{ab} = a_1b_1 + a_2b_2$. Here in (7)–(9) $x_{\mathbf{a}} \in \mathfrak{g}_{\mathbf{a}}$, $x_{\mathbf{b}} \in \mathfrak{g}_{\mathbf{b}}$, and the element $z \in \tilde{\mathfrak{g}}$ is not necessarily homogeneous. From (7) it follows that $\mathfrak{g}_{(0,0)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$, and the subspaces $\mathfrak{g}_{(1,1)}$, $\mathfrak{g}_{(1,0)}$, and $\mathfrak{g}_{(0,1)}$ are $\mathfrak{g}_{(0,0)}$-modules. It should be noted that $\mathfrak{g}_{(0,0)}\oplus\mathfrak{g}_{(1,1)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$ and the subspace $\mathfrak{g}_{(1,0)}\oplus\mathfrak{g}_{(0,1)}$ is a $\mathfrak{g}_{(0,0)}\oplus\mathfrak{g}_{(1,1)}$-module, and moreover $\{\mathfrak{g}_{(1,1)},\, \mathfrak{g}_{(1,0)}\} \subset \mathfrak{g}_{(0,1)}$ and vice versa $\{\mathfrak{g}_{(1,1)},\, \mathfrak{g}_{(0,1)}\} \subset \mathfrak{g}_{(1,0)}$. From (7) and (8) it follows that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements possesses two values: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$ as well as in the case of usual $\mathbb{Z}_2$-graded Lie superalgebras [9].

Now we construct a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded matrix superalgebras $\mathfrak{gl}(m_1,m_2|n_1,n_2)$.

Let an arbitrary $(m_1+m_2+n_1+n_2)\times(m_1+m_2+n_1+n_2)$-matrix M be presented in the following block form [1]:

$$M = \begin{pmatrix} A_{(0,0)} & A_{(1,1)} & A_{(1,0)} & A_{(0,1)} \\ B_{(1,1)} & B_{(0,0)} & B_{(0,1)} & B_{(1,0)} \\ C_{(1,0)} & C_{(0,1)} & C_{(0,0)} & C_{(1,1)} \\ D_{(0,1)} & D_{(1,0)} & D_{(1,1)} & D_{(0,0)} \end{pmatrix}, \tag{10}$$

where the diagonal block matrices $A_{(0,0)}, B_{(0,0)}, C_{(0,0)}, D_{(0,0)}$ have the dimensions $m_1\times m_1$, $m_2\times m_2$, $n_1\times n_1$, and $n_2\times n_2$, correspondingly, the dimensions of the nondiagonal block matrices $A_{(1,1)}, A_{(1,0)}, A_{(0,1)}$, etc., are easy determined by the dimensions of these diagonal block matrices. The matrix M can be split into the sum of four matrices:

$$\begin{aligned} M \;&=\; M_{(0,0)} + M_{(1,1)} + M_{(1,0)} + M_{(0,1)} \;= \\ &= \begin{pmatrix} A_{(0,0)} & 0 & 0 & 0 \\ 0 & B_{(0,0)} & 0 & 0 \\ 0 & 0 & C_{(0,0)} & 0 \\ 0 & 0 & 0 & D_{(0,0)} \end{pmatrix} + \begin{pmatrix} 0 & A_{(1,1)} & 0 & 0 \\ B_{(1,1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{(1,1)} \\ 0 & 0 & D_{(1,1)} & 0 \end{pmatrix} + \\ &+ \begin{pmatrix} 0 & 0 & A_{(1,0)} & 0 \\ 0 & 0 & 0 & B_{(1,0)} \\ C_{(1,0)} & 0 & 0 & 0 \\ 0 & D_{(1,0)} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & A_{(0,1)} \\ 0 & 0 & B_{(0,1)} & 0 \\ 0 & C_{(0,1)} & 0 & 0 \\ D_{(0,1)} & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{11}$$

Let us define the general commutator $[\![\cdot,\cdot]\!]$ on a space of all such matrices by the following way:

$$[\![M_{(a_1,a_2)}, M'_{(b_1,b_2)}]\!] := M_{(a_1,a_2)}M'_{(b_1,b_2)} - (-1)^{a_1b_1+a_2b_2}M'_{(b_1,b_2)}M_{(a_1,a_2)}, \tag{12}$$

[1] It is evidently supposed that all such matrices in each block-row or in each block-column have the same number of rows or columns.

for the homogeneous components $M_{(a_1,a_2)}$ and $M_{(b,b_2)}$. For arbitrary matrices M and M', the commutator $[\![\cdot,\cdot]\!]$ is extended by linearity. It is easy to check that

$$[\![M_{(a_1,a_2)}, M'_{(b_1,b_2)}]\!] = M''_{(a_1+a_2,b_1+b_2)}, \tag{13}$$

where the sum $(a_1 + a_2, b_1 + b_2)$ is defined $\mathrm{mod}\,(2,2)$. Thus the *grading* condition (7) is available. The *symmetry* and *Jacobi* identities (8) and (9) are available, too. Hence, we obtain a Lie superalgebra which is called $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. It should be noted that

$$\begin{aligned} [\![M_{\mathbf{a}}, M'_{\mathbf{b}}]\!] &= [M_{\mathbf{a}}, M'_{\mathbf{b}}], \quad \mathbf{ab} = 0, 2, \\ [\![M_{\mathbf{a}}, M'_{\mathbf{b}}]\!] &= \{M_{\mathbf{a}}, M'_{\mathbf{b}}\}, \quad \mathbf{ab} = 1. \end{aligned} \tag{14}$$

Now, we consider the Cartan–Weyl basis of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ and its supercommutation ($\mathbb{Z}_2 \times \mathbb{Z}_2$-graded) relations. In accordance with the block structure of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded matrix (10), we introduce a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded function (grading) $\mathbf{d}(\cdot)$ defined on the integer segment $[1, 2, \ldots, m_1, m_1 + 1, \ldots, m_1 + m_2, m_1 + m_2 + 1, \ldots, m_1 + m_2 + n_1, m_1 + m_2 + n_1 + 1, \ldots, m_1 + m_2 + n_1 + n_2]$ as follows:

$$\mathbf{d}_i \;:=\; \mathbf{d}(i) = \begin{cases} (0,0) & \text{for } i = 1, 2, \ldots, m_1, \\ (1,1) & \text{for } i = m_1 + 1, \ldots, m_1 + m_2, \\ (1,0) & \text{for } i = m_1 + m_2 + 1, \ldots, m_1 + m_2 + n_1, \\ (0,1) & \text{for } i = m_1 + m_2 + n_1 + 1, \ldots, m_1 + m_2 + n_1 + n_2. \end{cases} \tag{15}$$

Let e_{ij} be the $(m_1 + m_2 + n_1 + n_2) \times (m_1 + m_2 + n_1 + n_2)$ matrix (10) with 1 being in the (i, j)-th place and other entries 0. The matrices e_{ij} $(i, j = 1, 2, \ldots, m_1 + m_2 + n_1 + n_2)$ are homogeneous, moreover, the grading $\deg(e_{ij})$ is determined by

$$\deg(e_{ij}) = \mathbf{d}_{ij} := \mathbf{d}_i + \mathbf{d}_j \pmod{(2,2)}, \tag{16}$$

and the supercommutator for such matrices is given as follows:

$$[\![e_{ij},\, e_{kl}]\!] := e_{ij}e_{kl} - (-1)^{\mathbf{d}_{ij}\mathbf{d}_{kl}} e_{kl}e_{ij}. \tag{17}$$

It is easy to check that

$$[\![e_{ij},\, e_{kl}]\!] = \delta_{jk}e_{il} - (-1)^{\mathbf{d}_{ij}\mathbf{d}_{kl}} \delta_{il}e_{kj}. \tag{18}$$

The elements e_{ij} $(i, j = 1, 2, \ldots, m_1 + m_2 + n_1 + n_2)$ with the relations (18) generate the Lie superalgebra $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. The elements $h_i := e_{ii}$ $(i, j = 1, 2, \ldots, m_1 + m_2 + n_1 + n_2)$ compose a basis in the Cartan subalgebra $\mathfrak{h}(m_1 + m_2|n_1 + n_2) \subset \mathfrak{gl}(m_1, m_2|n_1, n_2)$.

The Lie superalgebra $\mathfrak{gl}(m_1, m_2|n_1, n_2$ plays a special role among all finite dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras. Namely, a general Ado's theorem is valid. It states: *Any finite dimensional Lie $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra can be realized in terms of a subalgebra of* $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. This theorem was proved by Scheunert [10] for all finite dimensional graded generalized Lie algebras including our cases.

As an illustration of Ado's theorem, in the next section we give realization of the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$ in terms of the superalgebra $\mathfrak{gl}(2m_1 + 1, 2m_2|2n, 0)$ and, moreover, we present a Cartan–Weyl basis of the orthosymplectic superalgebra and its explicit commutation relations, and we also show that a subset of the short root vectors of the Cartan–Weyl basis generates this superalgebra, and describe the parastatistics with the relative para-Fermi and para-Bose sets, simultaneously.

2. ORTHOSYMPLECTIC SUPERALGEBRA $\mathfrak{osp}(2m_1+1, 2m_2|2n, 0)$ AND ITS RELATION WITH PARASTATISTICS

We start with an explicit description of embedding of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m_1+1, 2m_2|2n, 0)$ in the general linear Lie superalgebra $\mathfrak{gl}(2m_1+1, 2m_2|2n, 0)$. For this propose, the $\mathbb{Z}_2\times\mathbb{Z}_2$-graded integer segment $\mathbb{S}_N^{(\mathbf{d})} := [1, 2, \ldots, 2N+1]$, where $N = m_1+m_2+n$, with the grading $d(\cdot)$ given by

$$\mathbf{d}_i \;:=\; \mathbf{d}(i) \;=\; \begin{cases} (0,0) & \text{for } i = 1, 2, \ldots, 2m_1, \\ (1,1) & \text{for } i = 2m_1+1, \ldots, 2m_1+2m_2, \\ (1,0) & \text{for } i = 2m_1+2m_2+1, \ldots, 2m_1+2m_2+2n, \end{cases} \tag{19}$$

is reindexed by the following way $\tilde{\mathbb{S}}_N^{(\mathbf{d})} := [0, \pm1, \pm2, \ldots, \pm N]$ with the grading $\mathbf{d}(\cdot)$ given by

$$\mathbf{d}_i \;:=\; \mathbf{d}(i) \;=\; \begin{cases} (0,0) & \text{for } i = 0, \pm1, \pm2, \ldots, \pm m_1, \\ (1,1) & \text{for } i = \pm(m_1+1), \ldots, \pm(m_1+m_2), \\ (1,0) & \text{for } i = \pm(m_1+m_2+1), \ldots, \pm(m_1+m_2+n). \end{cases} \tag{20}$$

Rows and columns of $\mathbb{Z}_2\times\mathbb{Z}_2$-graded $(2N+1)\times(2N+1)$-matrices are enumerated by the indices $0, 1, -1, 2, -2, \ldots, N, -N$ $(N = m_1+m_2+n)$. Let $e_{ij}(i, j \in \tilde{\mathbb{S}}_N^{(\mathbf{d})})$ be the standard (unit) basis of $\mathfrak{gl}(2m_1+1, 2m_2|2n, 0)$ with the given indexing and the canonical supercommutation relations:

$$[\![e_{ij},\, e_{kl}]\!] = \delta_{jk}e_{il} - (-1)^{\mathbf{d}_{ij}\mathbf{d}_{kl}}\delta_{il}e_{kj}, \tag{21}$$

where $\mathbf{d}_{ij} = \mathbf{d}_i + \mathbf{d}_j$, and the grading $\mathbf{d}(\cdot)$ is given by (20).

The orthosymplectic ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) Lie superalgebra $\mathfrak{osp}(2m_1+1, 2m_2|2n, 0)$ is embedded in $\mathfrak{gl}(2m_1+1, 2m_2|2n, 0)$ as a linear span of the elements

$$x_{ij} := e_{i,-j} - (-1)^{\mathbf{d}_i\mathbf{d}_j+\mathbf{d}_{ij}^2}\phi_i\phi_j e_{j,-i} \quad (i, j \in \tilde{\mathbb{S}}_N^{(\mathbf{d})}), \tag{22}$$

where the index function ϕ_i is given as follows:

$$\phi_i \;:=\begin{cases} 1 & \text{if } i = 0, \pm1, \pm2, \ldots, \pm(m_1+m_2), \\ 1 & \text{if } i = m_1+m_2+1, \ldots, m_1+m_2+n, \\ -1 & \text{if } i = -m_1-m_2-1, \ldots, -m_1-m_2-n. \end{cases} \tag{23}$$

It is easy to verify that elements (22) satisfy the following supercommutation relations:

$$\begin{aligned} [\![x_{ij},\, x_{kl}]\!] = \delta_{j,-k}x_{il} - \delta_{j,-l}(-1)^{\mathbf{d}_k\mathbf{d}_l+\mathbf{d}_{kl}^2}\phi_k\phi_l x_{ik} - \\ - \delta_{i,-k}(-1)^{\mathbf{d}_i\mathbf{d}_j+\mathbf{d}_{ij}^2}\phi_i\phi_j x_{jl} - \delta_{i,-l}(-1)^{\mathbf{d}_{ij}\mathbf{d}_{ik}}x_{kj}. \end{aligned} \tag{24}$$

Not all elements (22) are linearly independent because they satisfy the relations

$$x_{ij} = -(-1)^{\mathbf{d}_i\mathbf{d}_j+\mathbf{d}_{ij}^2}\phi_i\phi_j x_{ji} \quad (i, j \in \tilde{\mathbb{S}}_N^{(\mathbf{d})}), \tag{25}$$

and, what is more,

$$x_{ii} = 0 \quad \text{for} \;\; i = 0, \pm1, \pm2, \ldots, \pm(m_1 + m_2). \tag{26}$$

From the general supercommutation relations (24), it follows at once that the short root vectors x_{0i} $(i = \pm1, \pm2, \ldots, \pm(m_1 + m_2 + n))$ satisfy the following triple relations:

$$[\![[\![x_{0i},\, x_{0j}]\!], x_{0k}]\!] = -\delta_{j,-k}\phi_j x_{0i} + \delta_{i,-k}(-1)^{\mathbf{d}_i\mathbf{d}_j}\phi_i x_{0j}. \tag{27}$$

Conversely, let the abstract $\mathbb{Z}_2\times\mathbb{Z}_2$-graded generators x_{0i} $(i = \pm1, \pm2, \ldots, \pm(m_1+m_2+n))$ with the grading $\deg(x_{0i}) = \mathbf{d}_{0\mathrm{i}} \equiv \mathbf{d}_0 + \mathbf{d}_{\mathrm{i}} = \mathbf{d}_{\mathrm{i}}$, where $\mathbf{d}_{\mathrm{i}}$ is given by (20), satisfy the relations (27), where the index function ϕ_i is determined by (23), then it is not difficult to check that these relations generate for the superalgebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$.

The defining relations (27) can be rewritten in detail in accordance with the explicit grading of their generators using the following notations [1]:

$$\begin{aligned}
a_i^{-\zeta} &:= \zeta\sqrt{2}\,x_{0,\zeta i} && (\deg(a_i^{\zeta}) \;= (0,0)) \quad \text{for} \;\; i = 1, 2, \ldots, m_1,\\
\tilde{a}_i^{-\zeta} &:= \zeta\sqrt{2}\,x_{0,\zeta(m_1+i)} && (\deg(\tilde{a}_i^{\zeta}) = (1,1)) \quad \text{for} \;\; i = 1, 2, \ldots, m_2,\\
b_i^{-\zeta} &:= \zeta\sqrt{2}\,x_{0,\zeta(m+i)} && (\deg(b_i^{\zeta}) = (1,0)) \quad \text{for} \;\; i = 1, 2, \ldots, n,
\end{aligned} \tag{28}$$

where $m := m_1 + m_2$. Substituting (28) in (27), we obtain the different types of defining triple relations.

1. Parafermion relations:

(a1) the defining relations of $\mathfrak{o}(2m_1 + 1)$:

$$[[a_i^{\zeta}, a_j^{\eta}], a_k^{\xi}] = |\xi - \eta|\delta_{jk}a_i^{\zeta} - |\xi - \zeta|\delta_{ik}a_j^{\eta} \quad \text{for} \;\; i,j,k \;=\; 1, 2, \ldots, m_1; \tag{29}$$

(a2) the defining relations of $\mathfrak{o}(2m_2 + 1)$:

$$[[\tilde{a}_i^{\zeta}, \tilde{a}_j^{\eta}], \tilde{a}_k^{\xi}] = |\xi - \eta|\delta_{jk}\tilde{a}_i^{\zeta} - |\xi - \zeta|\delta_{ik}\tilde{a}_j^{\eta} \quad \text{for} \;\; i,j,k \;=\; 1, 2, \ldots, m_2; \tag{30}$$

(a3) the mixed parafermion relations:

$$[[a_i^{\zeta}, a_j^{\eta}], \tilde{a}_k^{\xi}] = 0, \qquad\qquad [[\tilde{a}_i^{\zeta}, \tilde{a}_j^{\eta}], a_k^{\xi}] = 0, \tag{31}$$

$$[[a_i^{\zeta}, \tilde{a}_j^{\eta}], a_k^{\xi}] = -|\xi - \zeta|\delta_{ik}\tilde{a}_j^{\eta}, \qquad [[a_i^{\zeta}, \tilde{a}_j^{\eta}], \tilde{a}_k^{\xi}] = |\xi - \eta|\delta_{jk}a_i^{\zeta}, \tag{32}$$

where $i,j,k = 1, \ldots, m_1$ for the symbols a's and $i,j,k = 1, \ldots, m_2$ for the symbols b's.

2. Paraboson relations:

(b1) the defining relations of $\mathfrak{osp}(1|2n_1)$:

$$[\{\tilde{b}_i^{\zeta}, \tilde{b}_j^{\eta}\}, \tilde{b}_k^{\xi}] = (\xi - \eta)\delta_{jk}\tilde{b}_i^{\zeta} + (\xi - \zeta)\delta_{ik}\tilde{b}_j^{\eta} \quad \text{for} \;\; i,j,k \;=\; 1, 2, \ldots, n. \tag{33}$$

[1] Here anywhere $\zeta, \eta, \xi \in \{+, -\}$.

3. Mixed parafermion and paraboson relations:

(ab1) the relative para-Fermi set:

$$\begin{aligned} &[[a_i^\zeta, a_j^\eta], b_k^\xi] = 0, && [\{b_i^\zeta, b_j^\eta\}, a_k^\xi] = 0, \\ &[[a_i^\zeta, b_j^\eta], a_k^\xi] = -|\xi - \zeta|\delta_{ik} b_j^\eta, && \{[a_i^\zeta, b_j^\eta], b_k^\xi\} = (\xi - \eta)\delta_{jk} a_i^\zeta, \end{aligned} \tag{34}$$

where $i, j, k = 1, 2, \ldots, m_1$ for the symbols a's and $i, j, k = 1, \ldots, n$ for the symbols b's;

(ab2) the relative para-Bose set:

$$\begin{aligned} &[[\tilde{a}_i^\zeta, \tilde{a}_j^\eta], b_k^\xi] = 0, && [\{b_i^\zeta, b_j^\eta\}, \tilde{a}_k^\xi] = 0, \\ &\{\{\tilde{a}_i^\zeta, b_j^\eta\}, \tilde{a}_k^\xi\} = |\xi - \zeta|\delta_{ik} b_j^\eta, && [\{\tilde{a}_i^\zeta, b_j^\eta\}, b_k^\xi] = (\xi - \eta)\delta_{jk} \tilde{a}_i^\zeta, \end{aligned} \tag{35}$$

where $i, j, k = 1, 2, \ldots, m_2$ for the symbols $\tilde{a}$'s and $i, j, k = 1, \ldots, n$ for the symbols b's;

(ab3) the relations with distinct grading elements:

$$\{[a_i^\zeta, \tilde{a}_j^\eta], b_k^\xi\} = [\{\tilde{a}_j^\eta, b_k^\xi\}, a_i^\zeta] = \{[b_k^\xi, a_i^\zeta], \tilde{a}_j^\eta\} = 0. \tag{36}$$

The result connected with relation (27) can be reformulated in the following way. *If we have two sorts of the parafermions a_i^ζ ($i = 1, 2, \ldots, m_1$) and $\tilde{a}_i^\zeta$ ($i = 1, 2, \ldots, m_2$) with the triple relations (29)–(32) and one sort of the parabosons b_i^ζ ($i = 1, 2, \ldots, n$) with the triple relations (33) which together satisfy the relative para-Fermi set (34) and relative para-Bose set (35), and they obey also the triple relations of the form (36), then this parasystem generates the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$.*

We consider two particular cases which are degenerations of $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$.

• If the parasystem consists of only one sort of the parafermions a_i^ζ ($i = 1, 2, \ldots, m_1$) and one sort of the parabosons b_i^ζ ($i = 1, 2, \ldots, n$), then we have the parasystem with the relative Fermi set and it generates the orthosymplectic $\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m_1{+}1|2n_1) = \mathfrak{osp}(2m_1 + 1, 0|2n_1, 0)$.

• If the parasystem contains one sort of the parafermions $\tilde{a}_i^\zeta$ ($i = 1, 2, \ldots, m_2$) and one sort of the parabosons b_i^ζ ($i = 1, 2, \ldots, n_1$), then we have the case of a parasystem with the relative Bose set (see the relations (2), (3) and (5)), and it generates the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1, 2m_2|2n, 0)$.

Thus we shown that the para-Fermi and para-Boose triple relations (2), (3) together with the relative para-Bose set (5) generate the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1, 2m|2n, 0)$. Moreover, it was shown that the superalgebras $\mathfrak{osp}(m_1, 2m_2|2n, 0)$ give more complex para-Fermi and para-Bose system which contains the relative para-Fermi and para-Bose sets, simultaneously.

It should be noted that, probably, for the first time the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded structure of the relative para-Bose set (5) was observable in [11, 12] (also see [13]).

It should be also noted that the obtained relation between the parastatistics and the orthosymplectic superalgebras allows one to apply all mathematical power of the representation theory of the superalgebras for a detailed description of the parastatistics, e.g., their Fock spaces, etc. (for example, see [14–16]).

Acknowledgements. The author would like to thank the Organizers for the kind invitation to speak at the International Workshop on Supersymmetries and Quantum Symmetries, Dubna, July 29 – August 3, 2013. The paper was supported by the RFBR grant No. 11-01-00980-a and the grant No. 12-09-0064 of the Academic Fund Program of the National Research University Higher School of Economics.

REFERENCES

1. *Green H. S.* A Generalized Method of Field Quantization // Phys. Rev. 1953. V. 9. P. 270–273.
2. *Kamefuchi S., Takahashi Y.* A Generalization of Field Quantization and Statistics // Nucl. Phys. 1962. V. 36. P. 177–206.
3. *Ryan C., Sudarshan E. C. G.* Representations of Para-Fermi Rings // Nucl. Phys. 1963. V. 47. P. 207–211.
4. *Ganchev A. C., Palev T. D.* A Lie Superalgebraic Interpretation of the Para-Bose Statistics // J. Math. Phys. 1980. V. 21. P. 797–799.
5. *Greenberg O. W., Messiah A. M. L.* Selection Rules for Parafields and the Absence of Paraparticles in Nature // Phys. Rev. B. 1965. V. 138. P. 1155–1167.
6. *Palev T. D.* Para-Bose and Para-Fermi Operators as Generators of Orthosymplectic Lie Superalgebras // J. Math. Phys. 1982. V. 23. P. 1100–1102.
7. *Rittenberg V., Wyler D.* Sequences of $\mathbb{Z}_2 \times \mathbb{Z}_2$-Graded Lie Algebras and Superalgebras // J. Math. Phys. 1978. V. 19, No. 10. P. 2193–2200.
8. *Rittenberg V., Wyler D.* Generalized Superalgebras // Nucl. Phys. B. 1978. V. 139, No. 10. P. 189–202.
9. *Kac V. G.* Lie Superalgebras // Adv. Math. 1977. V. 26. P. 8–96.
10. *Scheunert M.* Generalized Lie Algebras // J. Math. Phys. 1979. V. 20, No. 4. P. 712–720.
11. *Yang W., Jing S.* A New Kind of Graded Lie Algebra and Parastatistical Supersymmetry // Science in China. Ser. A. 2001. V. 44, No. 9. P. 1167–1173; arXiv:math-ph/0212004v1.
12. *Yang W., Jing S.* Graded Lie Algebra Generating of Parastatistical Algebraic Structure // Commun. Theor. Phys. 2001. V. 36. P. 647–665; arXiv:math-ph/0212009v1.
13. *Kanakoglou K., Daskaloyannis C., Herrera-Aguilar A.* Mixed Paraparticles, Colors, Braidings and a New Class of Realizations for Lie Superalgebras. arXiv:0912.1070v1 [math-ph]. 2009. P. 1–19.
14. *Lievens S., Stoilova N. I., Van der Jeugt J.* The Paraboson Fock Space and Unitary Irreducible Representations of the Lie Superalgebra $\mathfrak{osp}(1|2n)$ // Commun. Math. Phys. 2008. V. 281. P. 805–826; arXiv:0706.4196v2 [hep-th].
15. *Stoilova N. I., Van der Jeugt J.* The Parafermion Fock Space and Explicit $\mathfrak{so}(2n+1)$ Representations // J. Phys. A. 2008. V. 41. P. 075202-1–075202-13; arXiv:0712.1485v1 [hep-th].
16. *Stoilova N. I.* The Parastatistics Fock Space and Explicit Lie Superalgebra Representations. arXiv:1311.4042v1 [math-ph]. 2013. P. 1–14.