Письма в ЭЧАЯ. 2014. Т. 11, № 7(191). С. 1457–1461

ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

EXTREMAL VECTORS OF THE VERMA MODULES OF THE LIE ALGEBRA B_2 IN POINCARÉ–BIRKHOFF–WITT BASIS

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The aim of this work is to present a full set of expressions for the extremal vectors in a Verma module over the B_2 complex semisimple Lie algebra in the Poincaré–Birkhoff–Witt basis.

PACS: 02.20.Sv

INTRODUCTION

Verma modules were first considered by Verma in [1]. His study was continued by I. N. Berstein, I. M. Gelfand, and S. I. Gelfand [2]. Verma modules are modules over complex semisimple Lie algebras and they are useful in the representation theory and mathematical physics. The monograph [3] provides a thorough exposition of the Verma modules theory.

Extremal vectors of Verma modules appeared in the representation theory of the semisimple Lie algebras and group [2–4]. In [5], root systems of all types are considered, and the solution is given relatively to the so-called straight roots by using a special basis of the universal enveloping algebra. In [6,7], the solution is given for types A_n and D_n for all roots in the Poincaré–Birkhoff–Witt basis.

Our approach is a little different. We first construct some boson realization [8,9] and then transfer the problem to the problem of finding polynomial solutions of a system of differential equation.

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1. LIE ALGEBRA B_2 AND EXTREMAL VECTORS OF THE VERMA MODULES

The Cartan matrix of the Lie algebra B_2 is $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. The simple positive roots are $\boldsymbol{\alpha}_1 = (2, -2)$ and $\boldsymbol{\alpha}_2 = (-1, 2)$. We denote $\mathbf{E}_i = \mathbf{X}_{\boldsymbol{\alpha}_i}$ and $\mathbf{F}_i = \mathbf{X}_{-\boldsymbol{\alpha}_i}$, where $\mathbf{X}_{\boldsymbol{\alpha}}$ is the element of the Lie algebra which corresponds to the root $\boldsymbol{\alpha}$. We choose the Poincaré–Birkhoff–Witt basis consisting of elements $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{E}_k$, and $\mathbf{F}_k$, where $k = 1, \ldots, 4$, which are defined by means of the relations

$$\big[\mathbf{E}_i, \mathbf{F}_i\big] = \mathbf{H}_i, \quad \big[\mathbf{H}_i, \mathbf{E}_i\big] = 2\mathbf{E}_i, \quad \big[\mathbf{H}_i, \mathbf{F}_i\big] = -2\mathbf{F}_i,$$

$$\mathbf{E}_3 = \big[\mathbf{E}_1, \mathbf{E}_2\big], \quad \mathbf{E}_4 = \frac{1}{2}\big[\mathbf{E}_2, \mathbf{E}_3\big], \quad \big[\mathbf{X}_{\boldsymbol{\alpha}}, \mathbf{X}_{\boldsymbol{\beta}}\big] = N_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbf{X}_{\boldsymbol{\alpha}+\boldsymbol{\beta}},$$

where $i = 1, 2$, $N_{-\boldsymbol{\alpha},-\boldsymbol{\beta}} = -N_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and $\mathbf{F}_3 = \mathbf{X}_{-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2}$, $\mathbf{F}_4 = \mathbf{X}_{-\boldsymbol{\alpha}_1-2\boldsymbol{\alpha}_2}$.

The basis of the Cartan subalgebra $\mathfrak{h}$ is formed by the elements $\mathbf{H}_1$ and $\mathbf{H}_2$. Let us denote by $\mathfrak{n}_+$ and $\mathfrak{n}_-$ the Lie algebra generated by elements $\mathbf{E}_k$, $\mathbf{F}_k$, respectively, where $k = 1, \ldots, 4$, and $\mathfrak{b}_+ = \mathfrak{h} + \mathbf{n}_+$.

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{h}^*$, where $\lambda_1 = \boldsymbol{\lambda}(\mathbf{H}_1)$, $\lambda_2 = \boldsymbol{\lambda}(\mathbf{H}_2) \in \mathbb{C}$. Let us consider the one-dimensional representation $\tau_{\boldsymbol{\lambda}}$ of the Lie algebra $\mathfrak{b}_+$ for which

$$\tau_{\boldsymbol{\lambda}}(\mathbf{H} + \mathbf{E})|0\rangle = \boldsymbol{\lambda}(\mathbf{H})|0\rangle$$

holds for any $\mathbf{H} \in \mathfrak{h}$ and $\mathbf{E} \in \mathfrak{n}_+$. The element $|0\rangle$ will be called the highest-weight vector. Let us consider

$$W(\boldsymbol{\lambda}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}|0\rangle,$$

where $\mathfrak{b}_+$-module $\mathbb{C}|0\rangle$ is defined by the representation $\tau_{\boldsymbol{\lambda}}$. The space $W(\boldsymbol{\lambda}) \sim U(\mathfrak{n}_-)|0\rangle$ is $U(\mathfrak{g})$-module for the left regular representation which we call Verma module [1].

It is known [3] that any $U(\mathfrak{g})$-submodule of the module $W(\boldsymbol{\lambda})$ is isomorphic to the module $W(\boldsymbol{\mu})$, where

$$\boldsymbol{\mu} = \boldsymbol{\lambda} - n_1\boldsymbol{\alpha}_1 - n_2\boldsymbol{\alpha}_2, \quad n_1,\, n_2 \in \mathbb{N}_0 = \{0,\, 1,\, 2,\, \ldots\}.$$

Then for the highest-weight vector $|0\rangle_{\boldsymbol{\mu}}$ of the module $W(\boldsymbol{\mu}) \subset W(\boldsymbol{\lambda})$

$$\mathbf{H}|0\rangle_{\boldsymbol{\mu}} = \boldsymbol{\mu}(\mathbf{H})|0\rangle_{\boldsymbol{\mu}}, \quad \mathbf{H} \in \mathfrak{h}, \quad \mathbf{E}|0\rangle_{\boldsymbol{\mu}} = 0, \quad \mathbf{E} \in \mathfrak{n}_+$$

hold. The vectors $|0\rangle_{\boldsymbol{\mu}}$ are called the extremal (or singular) vectors of the module $W(\boldsymbol{\lambda})$. We find all extremal vectors of the Verma modules for the Lie algebra B_2 explicitly.

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[1] In Dixmiér [3], the Verma module $M(\boldsymbol{\lambda})$ is defined by the representation $\tau_{\boldsymbol{\lambda}-\boldsymbol{\delta}}$, where $\boldsymbol{\delta} = \frac{1}{2}\sum\limits_{k=1}^{4}\boldsymbol{\alpha}_k = (1, 1)$. So $W(\boldsymbol{\lambda}) = M(\boldsymbol{\lambda} + \boldsymbol{\delta})$.

2. DIFFERENTIAL EQUATIONS FOR EXTREMAL VECTORS

Let $\lambda_1,\ \lambda_2 \in \mathbb{C}$. The basis of the Verma module is formed by the vectors

$$|\mathbf{n}\rangle = |n_1, n_3, n_4, n_2\rangle = \mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle, \quad n_1,\ n_2,\ n_3,\ n_4 \in \mathbb{N}_0.$$

It is easy to show by direct calculations that it is true

$$\mathbf{H}_1|\mathbf{n}\rangle = (\lambda_1 - 2n_1 + n_2 - n_3)|\mathbf{n}\rangle, \quad \mathbf{H}_2|\mathbf{n}\rangle = (\lambda_2 + 2n_1 - 2n_2 - 2n_4)|\mathbf{n}\rangle,$$

$$\mathbf{E}_1|\mathbf{n}\rangle = n_1(\lambda_1 - n_1 + n_2 - n_3 - 1)|n_1 - 1, n_3, n_4, n_2\rangle - \\ - n_3|n_1, n_3 - 1, n_4, n_2 + 1\rangle + n_3(n_3 - 1)|n_1, n_3 - 2, n_4 + 1, n_2\rangle,$$

$$\mathbf{E}_2|\mathbf{n}\rangle = n_2(\lambda_2 - n_2 + 1)|n_1, n_3, n_4, n_2 - 1\rangle + \\ + 2n_3|n_1 + 1, n_3 - 1, n_4, n_2\rangle - n_4|n_1, n_3 + 1, n_4 - 1, n_2\rangle.$$

If we put

$$|n_1, n_3, n_4, n_2\rangle = \mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle \ \leftrightarrow\ z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4},$$

we see that the representation can be realized on the space of polynomials $f(z_1, z_2, z_3, z_4)$ of the variables z_1, z_2, z_3, and z_4. The equations for extremal vectors follow the system of partial differential equations

$$\begin{aligned} \lambda_1 f - 2z_1 f_1 + z_2 f_2 - z_3 f_3 &= \mu_1 f, \\ \lambda_2 f + 2z_1 f_1 - 2z_2 f_2 - 2z_4 f_4 &= \mu_2 f, \\ \lambda_1 f_1 - z_1 f_{11} + z_2 f_{12} - z_3 f_{13} - z_2 f_3 + z_4 f_{33} &= 0, \\ \lambda_2 f_2 - z_2 f_{22} + 2z_1 f_3 - z_3 f_4 &= 0. \end{aligned} \tag{1}$$

3. EXTREMAL VECTORS

It is evident that the extremal vectors are in the one-to-one correspondence with polynomial solution of the system (1). By the standard method it is possible to show that nonzero solution of this system exists only in eight cases.

In this section we will give the extremal vectors and the corresponding functionals $\boldsymbol{\mu}$.

1. $\boldsymbol{\mu} = (\lambda_1, \lambda_2)$, where $\lambda_1,\ \lambda_2 \in \mathbb{C}$. For any $\boldsymbol{\lambda}$ there is the trivial extremal vector $|0\rangle_{\boldsymbol{\mu}} = |0\rangle$. This extremal vector is the height-weight vector of the Verma module $W(\boldsymbol{\lambda})$.

2. $\boldsymbol{\mu} = (-\lambda_1 - 2, 2\lambda_1 + \lambda_2 + 2)$, where $\lambda_1 + 1 \in \mathbb{N}_0$ a $\lambda_2 \in \mathbb{C}$. In this case we obtain the extremal vector $|0\rangle_{\boldsymbol{\mu}} = \mathbf{F}_1^{\lambda_1+1}|0\rangle$.

3. $\boldsymbol{\mu} = (\lambda_1 + \lambda_2 + 1, -\lambda_2 - 2)$, where $\lambda_2 + 1 \in \mathbb{N}_0$ a $\lambda_1 \in \mathbb{C}$. The extremal vector in this case is $|0\rangle_{\boldsymbol{\mu}} = \mathbf{F}_2^{\lambda_2+1}|0\rangle$.

4. $\boldsymbol{\mu} = (-\lambda_1 - \lambda_2 - 3, 2\lambda_1 + \lambda_2 + 2)$, where $\lambda_1 + \lambda_2 + 2 \in \mathbb{N}_0$ and $\lambda_2 + 1 \in \mathbb{N}_0$. This case leads to the extremal vector $|0\rangle_{\boldsymbol{\mu}} = \mathbf{F}_1^{\lambda_1+\lambda_2+2}\mathbf{F}_2^{\lambda_2+1}|0\rangle$.

5. $\boldsymbol{\mu} = (\lambda_1 + \lambda_2 + 1, -2\lambda_1 - \lambda_2 - 4)$, where $\lambda_1 + 1 \in \mathbb{N}_0$ and $2\lambda_1 + \lambda_2 + 3 \in \mathbb{N}_0$. In this case we obtain the extremal vector

$$|0\rangle_{\boldsymbol{\mu}} = \sum_{n \leqslant k \leqslant 2n} \frac{(-\lambda_1 - 1)_n \, (-2\lambda_1 - \lambda_2 - 3)_k}{(k-n)! \, (2n-k)!} \, \mathbf{F}_1^{\lambda_1 - n + 1} \mathbf{F}_3^{2n-k} \mathbf{F}_4^{k-n} \mathbf{F}_2^{2\lambda_1 + \lambda_2 - k + 3} |0\rangle,$$

where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n \in \mathbb{N}$.

6. $\boldsymbol{\mu} = (-\lambda_1 - \lambda_2 - 3, \lambda_2)$, where $2\lambda_1 + \lambda_2 + 3 \in \mathbb{N}_0$. Extremal vector in this case is

$$|0\rangle_{\boldsymbol{\mu}} = \sum_{n \leqslant k \leqslant 2n} \frac{(-\lambda_1 - 1)_n \, (-2\lambda_1 - \lambda_2 - 3)_k}{(k-n)! \, (2n-k)!} \, \mathbf{F}_1^{2\lambda_1 + \lambda_2 - n + 3} \mathbf{F}_3^{2n-k} \mathbf{F}_4^{k-n} \mathbf{F}_2^{2\lambda_1 + \lambda_2 - k + 3} |0\rangle.$$

7. $\boldsymbol{\mu} = (\lambda_1, -2\lambda_1 - \lambda_2 - 4)$, where $\lambda_1 + \lambda_2 + 2 \in \mathbb{N}_0$. In this case we have the extremal vector

$$|0\rangle_{\boldsymbol{\mu}} = \sum_{n \leqslant k \leqslant 2n} \frac{(-\lambda_1 - \lambda_2 - 2)_n \, (-2\lambda_1 - \lambda_2 - 3)_k}{(k-n)! \, (2n-k)!} \, \mathbf{F}_1^{\lambda_1 + \lambda_2 - n + 2} \mathbf{F}_3^{2n-k} \mathbf{F}_4^{k-n} \mathbf{F}_2^{2\lambda_1 + 2\lambda_2 - k + 4} |0\rangle.$$

8. $\boldsymbol{\mu} = (-\lambda_1 - 2, -\lambda_2 - 2)$, where $2\lambda_1 + \lambda_2 + 3 \in \mathbb{N}_0$ and $2\lambda_1 + 2\lambda_2 + 4 \in \mathbb{N}_0$. The extremal vector is

$$|0\rangle_{\boldsymbol{\mu}} = \sum_{n \leqslant k \leqslant 2n} \frac{(-\lambda_1 - \lambda_2 - 2)_n \, (-2\lambda_1 - \lambda_2 - 3)_k}{(k-n)! \, (2n-k)!} \, \mathbf{F}_1^{2\lambda_1 + \lambda_2 - n + 3} \mathbf{F}_3^{2n-k} \mathbf{F}_4^{k-n} \mathbf{F}_2^{2\lambda_1 + 2\lambda_2 - k + 4} |0\rangle.$$

4. CONNECTION OF THE EXTREMAL VECTORS WITH THE WEYL GROUP

It is known, see, e.g., [3], that $W(\boldsymbol{\mu}) \subset W(\boldsymbol{\lambda})$ iff $\boldsymbol{\mu} = \boldsymbol{\lambda} - n_1\boldsymbol{\alpha}_1 - n_2\boldsymbol{\alpha}_2$, where $n_1, n_2 \in \mathbb{N}_0$, and there is an element w of the Weyl group $\mathfrak{W}$ such that

$$w(\boldsymbol{\lambda} + \boldsymbol{\delta}) = \boldsymbol{\mu} + \boldsymbol{\delta} = \boldsymbol{\lambda} - n_1\boldsymbol{\alpha}_1 - n_2\boldsymbol{\alpha}_2 + \boldsymbol{\delta}.$$

The above found extremal vectors correspond to the eight elements of the Weyl group.

The extremal vectors of the Verma modules have been intensively studied by Dobrev. He found in [5] some extremal vectors for the quantum groups $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra. Unlike this paper, he used the Cartan–Weyl generators of the quantum group.

Dobrev gives only the extremal vectors which correspond to the element of the Weyl group of the form $w = s_{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}$ is a root. His extremal vectors are, after transformation to the Poincaré–Birkhoff–Witt basis, the same as ours.

The extremal vectors corresponding to our cases 4, 5, and 8 are not included in the cited work. On the other hand, it is easy to see that these extremal vectors can be easily obtained from the extremal vectors, which correspond to these roots. For example in our case 4, we have $\lambda_2 + 1 \in \mathbb{N}_0$. Therefore the vector $|0\rangle_3 = \mathbf{F}_2^{\lambda_2+1}|0\rangle$ is extremal and also the highest-weight vector for Verma module with weight $\boldsymbol{\mu} = (\mu_1, \mu_2) = (\lambda_1 + \lambda_2 + 1, -\lambda_2 - 1)$. However in case 4, we have $\lambda_1 + \lambda_2 + 2 = \mu_1 + 1 \in \mathbb{N}_0$. So this Verma module has the extremal vector

$$\mathbf{F}_1^{\mu_1+1}|0\rangle_3 = \mathbf{F}_1^{\lambda_1+\lambda_2+2}\mathbf{F}_2^{\lambda_2+1}|0\rangle,$$

which is our extremal vector in case 4. Similarly, we can rewrite $|0\rangle_5 = \mathbf{F}_2^{2\lambda_1+\lambda_2+3}\mathbf{F}_1^{\lambda_1+1}|0\rangle$, $|0\rangle_8 = \mathbf{F}_1^{\lambda_1+1}|0\rangle_7$ for $\lambda_1 + 1 \in \mathbb{N}_0$, and for $\lambda_2 + 1 \in \mathbb{N}_0$ we can write the extremal vector $|0\rangle_8 = \mathbf{X}_6|0\rangle_3 = \mathbf{X}_6\mathbf{F}_2^{\lambda_2+1}|0\rangle$, where $\mathbf{X}_6$ is the element of $U(B_2)$ which corresponds to the extremal vector in our case 6.

The main aim of our paper was to give a new method, how to find the extremal vectors in the Poincaré–Birkhoff–Witt basis which are more frequently used in physics. We did not use from the beginning the Gelfand result. If we used this result, the solution would be much easier.

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