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ФИЗИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ И АТОМНОГО ЯДРА. ТЕОРИЯ

A HIGHER-SPIN CHERN–SIMONS THEORY OF ANYONS

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We propose Chern–Simons models of fractional-spin fields interacting with ordinary tensorial higher-spin fields and internal color gauge fields. For integer and half-integer values of the fractional spins, the model reduces to finite sets of fields modulo infinite-dimensional ideals. We present the model on-shell using Fock-space representations of the underlying deformed-oscillator algebra.

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Quantum mechanics in $2+1$ dimensions admits anyons: fractional-spin particles [1] with correlated generalized statistics properties governed by the braid group [2]. Anyons arise in a number of systems; for example, as non-relativistic charged vortices [3], relativistic Hopf-interacting massive particles in matter-coupled Chern–Simons theories [4, 5] and vertex operators in two-dimensional conformal field theories [6]. In this letter, we present topological models of Chern–Simons type that describe fractional-spin gauge fields coupled to higher-spin gravities (HSGRA) and internal gauge fields, which we will refer to as fractional-spin gravities. Diffeomorphism invariance as well as higher-spin symmetries are indeed natural in theories of anyons, essentially due to the topological character of the local rotations and translations underlying the generalized spin-statistics relations [2] and the fact that local constructs built out of fractional-spin fields decompose under the Lorentz algebra into infinite towers of higher-spin Lorentz tensors and tensor-spinors.

Our construction stands on the observation that the Prokushkin–Vasiliev system [7], which provides the only known fully non-linear description of three-dimensional matter-coupled HSGRA, admits several inequivalent embeddings of the Lorentz algebra into its higher-spin algebra [8], besides the standard embedding leading to tensor-spinorial HSGRA. The Prokushkin–Vasiliev system consists of a connection one-form $\widehat{A}$ and matter zero-form $\widehat{B}$ living on a base manifold given locally by the direct product of a commutative spacetime $\mathcal{M}$ and non-commutative twistor space $\mathcal{Z}$ with a closed and central two-form $\widehat{J}$. These master fields are valued in associative algebras consisting of functions on a fiber manifold $\mathcal{Y}\times\mathcal{I}$, the product of an additional twistor space $\mathcal{Y}$ and an internal manifold $\mathcal{I}$ whose coordinates generate a matrix algebra; for further details, see [8]. The Prokushkin–Vasiliev field equations, viz.

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$\widehat{d}\widehat{A}+\widehat{A}^2+\widehat{J}\widehat{B}=0$ and $\widehat{d}\widehat{B}+[\widehat{A},\widehat{B}]=0$, state that $\widehat{A}=\widehat{A}|_{\mathcal{M}}+\widehat{A}|_{\mathcal{Z}}$ describes a flat connection on $\mathcal{M}$ and a pair of oscillators on $\mathcal{Z}\times\mathcal{Y}$ deformed by local as well as topological degrees of freedom contained in $\widehat{B}$. The latter can acquire anti-de Sitter (AdS_3) vacuum expectation values, viz. $\langle\widehat{B}\rangle=\nu$. Truncating $\widehat{B}=\nu$ yields Chern–Simons-like HSGRAs on $\mathcal{M}$ of the type introduced originally in [9–11]. In the simplest set up, the flat master connection one-form

$$A:=\widehat{A}|_{\mathcal{M}}=\frac{1}{4i}\sum_{n\geqslant 0;s,t=0,1}A^{\alpha_1\cdots\alpha_n}_{s,t}\,\Gamma^s\,k^t\,q_{(\alpha_1}\cdots q_{\alpha_n)}\in Aq(2;\nu)\otimes\mathcal{C}\ell_1 \tag{1}$$

takes its values in the direct product of the enveloping algebra $Aq(2,\nu)$ of the Wigner-deformed canonical coordinates q_α ($\alpha=1,2$) and Kleinian k [11–13], viz.

$$[q_\alpha,q_\beta]=2i\epsilon_{\alpha\beta}(1+\nu k),\quad \{q_\alpha,k\}=0,\quad k^2=1, \tag{2}$$

and the Clifford algebra $\mathcal{C}\ell_1\cong\mathbb{Z}_2$ generated by a single bosonic element Γ obeying $\Gamma^2=1$. The gauge algebra contains several inequivalent AdS_3 sub-algebras. The standard choice

$$M^{(\mathrm{St})}_{\alpha\beta}:=q_{(\alpha}q_{\beta)},\quad P^{(\mathrm{St})}_{\alpha\beta}:=\Gamma q_{(\alpha}q_{\beta)} \tag{3}$$

yields tensor-spinorial HSGRAs of Chern–Simons type whose CFT duals [14] consist of quantum states with Bose or Fermi statistics, just as in $3+1$ dimensions [15]. On the other hand, the non-standard choice

$$M^{(\text{non-St})}_{\alpha\beta}:=\Pi_+\,q_{(\alpha}q_{\beta)},\quad P^{(\text{non-St})}_{\alpha\beta}:=\Gamma M^{(\text{non-St})}_{\alpha\beta},\quad \Pi_\pm:=\frac{1}{2}(1\pm k) \tag{4}$$

yields fractional-spin HSGRAs consisting of tensor-spinorial fields in $W:=\Pi_+\,A\,\Pi_+$ and Lorentz-singlet color gauge fields in $U:=\Pi_-\,A\,\Pi_-$ coupled to the bi-fundamental master fields $\psi:=\Pi_+\,A\,\Pi_-$ and $\bar\psi:=\Pi_-\,A\,\Pi_+$, which consist of fractional-spin fields in infinite-dimensional discrete-series representations of the Lorentz algebra, that can be either bosonic or fermionic [8]. Assembling the master fields into

$$\mathbb{A}=\begin{bmatrix} W & \psi \\ \overline{\psi} & U\end{bmatrix}, \tag{5}$$

it follows that $dA+A^2=0$ is equivalent to $d\mathbb{A}+\mathbb{A}^2=0$, i.e.,

$$dW+W\,W+\psi\,\overline{\psi}\;=\;0,\quad dU+U\,U+\overline{\psi}\,\psi\;=\;0, \tag{6}$$

$$d\psi+W\,\psi+\psi\,U\;=\;0,\quad d\overline{\psi}+\overline{\psi}\,W+U\,\overline{\psi}\;=\;0. \tag{7}$$

As for off-shell formulations, suitable bilinear forms are needed, such as the supertrace operation on $Aq(2,\nu)$ [11] used to ν-deform Blencowe's tensor-spinorial HSGRA theory [9]. To analyze the theory on-shell, it suffices, however, to use a representation of $Aq(2,\nu)$ in a standard Fock space $\mathcal{F}=\bigoplus_{n\geqslant 0}\mathbb{C}\otimes|n\rangle$ and its dual $\mathcal{F}^*=\bigoplus_{n\geqslant 0}\mathbb{C}\otimes\langle n|$ [13], viz.

$$a^\pm=\frac{1}{2}(q_1\mp iq_2),\quad a^\pm|_{\mathcal{F}}=\sum_{n\geqslant 0}\sqrt{[n+1]_\nu}\left|n+\frac{1}{2}(1\pm 1)\rangle\langle n+\frac{1}{2}(1\mp 1)\right|, \tag{8}$$

$$k|_{\mathcal{F}}=\sum_{n\geqslant 0}(-1)^n|n\rangle\langle n|,\quad [n]_\nu:=n+\frac{1}{2}(1-(-1)^n)\nu,\quad \langle n|n'\rangle=\delta_{n,n'}. \tag{9}$$

The master connection (1) is thus represented by

$$A|_{\mathcal{F}} = \frac{1}{4i} \sum_{m,n\geqslant 0; s=0,1} A_s^{m,n}\, \Gamma^s\, |m\rangle\langle n|, \quad A_s^{m,n} := \sum_{p\geqslant 0; t=0,1} A_{s,t}^{\alpha_1\cdots\alpha_p} (\mathcal{Q}_{\alpha_1\cdots\alpha_p}{}^t)^{m,n}, \quad (10)$$

where the matrix $(\mathcal{Q}_{\alpha_1\cdots\alpha_p}{}^t)^{m,n} := (-1)^{t\,m}\langle m|q_{(\alpha_1}\cdots q_{\alpha_p)}|n\rangle$ represents $Aq(2,\nu)$ in $\mathcal{F}$. Alternatively, using the notation of (5), the connection $\mathbb{A}|_{\mathcal{F}}$ takes the form

$$\left.\left[\begin{array}{cc} W & \psi \\ \overline{\psi} & U \end{array}\right]\right|_{\mathcal{F}} = \frac{1}{4i} \sum_{m,n\geqslant 0, s=0,1} \Gamma^s \left[\begin{array}{cc} W_s^{2m,2n}\,|2m\rangle\langle 2n| & \psi_s^{2m,2n+1}\,|2m\rangle\langle 2n+1| \\ \overline{\psi}_s^{2m+1,2n}\,|2m+1\rangle\langle 2n| & U_s^{2m+1,2n+1}\,|2m+1\rangle\langle 2n+1| \end{array}\right]. \quad (11)$$

The master fields W and U are thus represented faithfully in the even and odd Fock spaces $\Pi_+\mathcal{F}$ and $\Pi_-\mathcal{F}$, respectively. In particular, the Lorentz connection in W acts non-trivially in $\Pi_+\mathcal{F}$ that forms an irrep of the Lorentz algebra for generic ν, and acts trivially in $\Pi_-\mathcal{F}$. The $(\psi,\bar{\psi})$ fields exchange $\Pi_\pm\mathcal{F}$ and transform under one-sided higher-spin and internal gauge transformations. To compute the Lorentz spin of these intertwiners, one may first use Dirac matrices obeying $\{\gamma_a,\gamma_b\} = -2\eta_{ab}$, $\eta_{ab} = \mathrm{diag}\,(-1,+1,+1)$, to convert the standard embedding (3) of the Lorentz algebra into

$$J_a^{(\mathrm{St})} := \frac{i}{8}\epsilon^{\alpha\alpha'}(\gamma_a)_{\alpha'}{}^{\beta} M_{\alpha\beta}^{(\mathrm{St})} = \left(\frac{\{a^+,a^-\}}{4}, \frac{a^{+2}+a^{-2}}{2}, \frac{a^{+2}-a^{-2}}{2i}\right), \quad (12)$$

which acts non-trivially on $\Pi_\pm\mathcal{F}$. For the alternative embedding (4) of the Lorentz algebra, one then has

$$J_0^{(\text{non-St})}|_{\mathcal{F}} = \sum_{n\geqslant 0}\left(n + \frac{1}{4}(1+\nu)\right)|2n\rangle\langle 2n|, \quad (13)$$

$$J_\pm^{(\text{non-St})}|_{\mathcal{F}} = \sum_{n\geqslant 0}\sqrt{[2n+2]_\nu[2n+1]_\nu}\;|2n+1\pm 1\rangle\langle 2n+1\mp 1|, \quad (14)$$

where $J_\pm^{(\text{non-St})} := J_1^{(\text{non-St})} \pm \mathrm{i} J_2^{(\text{non-St})} = \frac{1}{2}\,\Pi_+(a^\pm)^2\Pi_+$, which indeed acts non-trivially on $\Pi_+\mathcal{F}$ while leaving $\Pi_-\mathcal{F}$ invariant. Thus, as for the quadratic Casimir operator, one has

$$C_2 := J^a J_a \;\Rightarrow C_2\psi|_{\mathcal{F}} = -s_\psi(s_\psi-1)\psi|_{\mathcal{F}}, \quad s_\psi = \frac{1}{4}(1+\nu), \quad (15)$$

that is, ψ has Lorentz spin $1/4(1+\nu)$, where ν can be any real number. For negative integer and negative half-integer lowest weights s_ψ, singular vectors arise in $\mathcal{F}$, viz.

$$a^-|2\ell+1\rangle = 0 = a^+|2\ell\rangle, \quad \nu = -2\ell-1, \quad \ell = 0,1,2,\ldots \quad (16)$$

Thus, for these values of ν, referred to as critical values, the representation of $Aq(2,\nu)$ in $\mathcal{F}$ decomposes into a finite-dimensional and an infinite-dimensional algebra as follows [11, 13]:

$$Aq(2;-2\ell-1)|_{\mathcal{F}} \cong gl(2\ell+1)\oplus Aq(2;2\ell+1)|_{\mathcal{F}}. \quad (17)$$

Thus, using Feigin's notation $gl(\lambda)$ [16], it follows that

$$W|_{\mathcal{F}} \in gl(-2s_\psi+1;\tau)\otimes \mathcal{C}\ell_1, \quad U|_{\mathcal{F}} \in gl(-2s_\psi;\tau)\otimes \mathcal{C}\ell_1, \tag{18}$$

which are infinite-dimensional algebras for generic s_ψ with critical limits given by semi-direct sums of a finite-dimensional and an infinite-dimensional sub-algebra with ideal structure controlled by τ, which can thus take three distinct values depending on whether one or the other or both of the sub-algebras are ideal; in the case at hand, τ is chosen in accordance with (17), that is, such that both sub-algebras are ideals; for further details, see [8]. Thus, the fractional-spin HSGRA model reduces to a certain conventional tensor-spinorial HSGRA in critical limits.

The representation of $f \in Aq(2,\nu)$ in $\mathcal{F}$ twists the Hermitian conjugation operation [8], viz. $(f^\dagger)|_{\mathcal{F}} \equiv ((CfC)|_{\mathcal{F}})^\dagger$ where $(q_\alpha)^\dagger := q_\alpha$, $k^\dagger := k$ and $(|m\rangle)^\dagger := \langle m|$ and the conjugation matrix obeys $C^2=1$ and $Ck=kC$ [13]. Taking $C^\dagger = C$, we can impose the reality condition

$$(\mathbb{A}|_{\mathcal{F}})^\dagger = -(\mathbb{C}\mathbb{A}\mathbb{C})|_{\mathcal{F}}, \quad \mathbb{C} = \left[\begin{array}{cc} \Pi_+ C & 0 \\ 0 & \Pi_- \end{array}\right], \tag{19}$$

for which U is valued in a compact real form of $\Pi_- Aq(2,\nu)\Pi_-$ represented unitarily in $\Pi_-\mathcal{F}$ for all ν, while W is valued in a non-compact real form of $\Pi_+ Aq(2,\nu)\Pi_+$ represented unitarily in $\Pi_+\mathcal{F}$ iff $\Pi_+ C = \Pi_+$, that is, iff $\nu \geqslant -1$. For $\nu < -1$ there are negative eigenvalues in $\Pi_+ C|_{\mathcal{F}}$ whose number grows linearly with $|\nu|$. Using the notation of analytically continued real forms [8, 16, 17], one has

$$W|_{\mathcal{F}} \in u(p_+,p_-;\tau)\otimes \mathcal{C}\ell_1, \quad U|_{\mathcal{F}} \in u(-2s_\psi;\tau)\otimes \mathcal{C}\ell_1, \tag{20}$$

$$p_+ + p_- = -2s_\psi + 1, \quad p_+ - p_- = 1 + (-1)^{-2s_\psi}. \tag{21}$$

For critical $\nu = -2\ell - 1$, one has $\Pi_+ C|_{\mathcal{F}} = \sum_{n=0}^{\ell}(-1)^n|2n\rangle\langle 2n| + \sum_{n\geqslant \ell+1}|2n\rangle\langle 2n|$ and hence

$$W|_{\mathcal{F}} \cap gl(2\ell+1) \in u(p_+,p_-)\otimes \mathbb{Z}_2, \quad U|_{\mathcal{F}} \cap gl(2\ell+1) \in u(\ell)\otimes \mathbb{Z}_2, \tag{22}$$

where $p_\pm = \frac{1}{2}(\ell + 1 \pm \frac{1+(-1)^\ell}{2})$, and $(\psi,\overline{\psi})|_{\mathcal{F}} \cap gl(2\ell+1)$ belong to bi-fundamental representations with integer or half-integer spins and finite-rank color indices. We note that if $\psi = |\sigma\rangle\langle c|$, where thus σ is a spin and c is a color, then $\bar\psi = -|c\rangle\langle\sigma|C$ and hence $\psi\bar\psi = |\sigma\rangle\langle\sigma|C$ while $\bar\psi\psi = |\sigma\rangle\langle c|C|c\rangle\langle\sigma|$ that can vanish in the non-unitary regime. Thus, the fractional-spin fields necessarily source the tensor-spinorial fields W (cf. positivity of energy in ordinary gravity), while the internal gauge field U can be truncated consistently leading to

$$dW + W^2 + \psi\bar\psi = 0, \quad d\psi + W\psi = 0, \quad d\bar\psi + \bar\psi W = 0, \quad \bar\psi\psi = 0, \tag{23}$$

which defines a quasi-free differential algebra. In summary, the model presented here may be of interest in the context of holography where $(\psi,\bar\psi)$ are expected to correspond to vertex operators with fractional conformal weights resulting in multi-valued correlation functions forming anyonic representations of the braid group, possibly along the lines of [6]. The model also serves as a starting point for incorporating local FS degrees of freedom in three and four dimensions; the latter may correspond holographically to massive anyons in three-dimensional quantum field theories. As for their quantization, we propose to use generalized Poisson sigma models [8]. Subjected to suitable boundary conditions, these sigma models may remain weakly coupled in the critical limits of the Prokushkin–Vasiliev system, which are otherwise strongly coupled limits of the standard Chern–Simons formulation.

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