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A UNIFIED MODE DECOMPOSITION METHOD FOR PHYSICAL FIELDS IN HOMOGENEOUS COSMOLOGY

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A general and holistic framework of mode decomposition and harmonic analysis is developed for vector valued fields in curved spacetimes.

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1. MOTIVATION

The theoretical framework of this study is the Quantum Field Theory in Curved Spacetimes (QFT in CST). The idea of this theory is that one considers gravity as being purely classical, given by the theory of General Relativity (GR). Following GR, the gravity is reflected in the curvature of the spacetime geometry. All other fields of nature are considered as relativistic quantum fields propagating in this curved spacetime. The back reaction of these fields on the gravity is given by the semiclassical Einstein's equation,

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi\langle T_{\mu\nu}\rangle_\omega,$$

where $G_{\mu\nu}$ is Einstein's tensor; $g_{\mu\nu}$ is the spacetime metric; Λ is the cosmological constant, and $\langle T_{\mu\nu}\rangle_\omega$ is the expectation value of the stress-energy tensor $T_{\mu\nu}$ of all quantum fields present in a given quantum state ω of the system. Such a semiclassical approach is justified in physical systems where the gravity is strong enough so that it influences the behavior of quantum fields substantially, and at the same time its possible quantum fluctuations are negligible, so that its coupling with quantum fields can be considered purely classical. Two prominent real-world situations, where such systems arise, are the vicinities of black holes and the early epoch of the universe. Being interested mainly in cosmology, we concentrate on the second case. One particular quantum effect, which arises solely due to the gravity, is the cosmological particle creation. A completely rigorous and explicit calculation of this effect has been carried out in [1] for the scalar field in Friedman–Robertson–Walker (FRW) spacetimes. The two main mathematical methods that underlie the chain of results which has led to these calculations are the mode decomposition and harmonic/Fourier analysis.

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To be able to produce results comparable with observations, we need to consider more realistic situations rather than scalar fields on FRW spacetimes. For this reason, we want to generalize the calculations performed in [1] to other cosmological models. To achieve this, we put forth the aim to extend the two methods, mode decomposition and Fourier analysis, in the following three directions:

- vector-valued fields (e.g., tensor, spinor);
- non-FRW spacetimes (e.g., purely homogeneous);
- weak (distributional) solutions.

This will allow us to encompass almost all cosmological models.

2. MODE DECOMPOSITION

We start by focusing on the mode decomposition method. Our geometrical setup includes an n-dimensional complex vector bundle $\mathcal{T} \xrightarrow{\pi} M$ over a 4-dimensional globally hyperbolic spacetime $M \sim \mathbb{R} \times \Sigma$. Global hyperbolicity basically amounts to saying that Cauchy problem for (strictly) hyperbolic PDE is well-posed. We further consider $\langle , \rangle_{\mathfrak{g}}$ — fiber metric on $\mathcal{T}$, ∇ — connection on $\mathcal{T}$ s.t. $\nabla \langle , \rangle_{\mathfrak{g}} = 0$. A (classical) field ϕ is a C^∞ solution of the field equation

$$D\phi(x) = \left(\Box^\nabla + m^*(x)\right) \phi(x) = 0,$$

where $m^*(x)$ is the smooth and real variable mass term. Consider $Sol_0(\mathcal{T})$ — solutions ϕ s.t. for every t, $\phi(t, .) \in C_0^\infty(\Sigma)$ (strong solutions).

Definition 1. *$Sol_0(\mathcal{T})$ allows for a mode decomposition iff $\exists (\mathfrak{M}, d\mathfrak{m})$ a measure space, and*

$$\{T_\alpha(t) \in C^\infty(\mathbb{R})\}_{\alpha \in \mathfrak{M}}, \quad \{X_\alpha(\mathbf{x}) \in C^\infty(\pi^{-1}(\Sigma))\}_{\alpha \in \mathfrak{M}}$$

measurable fields of functions s.t.

$$DT_\alpha(t)X_\alpha(\mathbf{x}) = D\bar{T}_\alpha(t)X_\alpha(\mathbf{x}) = 0,$$

$T_\alpha(t)$ and $\bar{T}_\alpha(t)$ linearly independent a.e. in $\mathfrak{M}$, and for any $\phi \in Sol_0(\mathcal{T})$ there are unique mode coefficients $a^\phi(\alpha), b^\phi(\alpha) \in L^1_{\text{loc}}(\mathfrak{M})$ with

$$\phi(x) = \int_{\mathfrak{M}} d\mathfrak{m}(\alpha) \left(a^\phi(\alpha) T_\alpha(t) X_\alpha(\mathbf{x}) + b^\phi(\alpha) \bar{T}_\alpha(t) X_\alpha(\mathbf{x})\right).$$

Proposition 1. *$Sol_0(\mathcal{T})$ can be mode decomposed iff there exists a covering of $\mathcal{T}$ by local trivializations s.t. everywhere holds:*

(i) $g_{00} = g_{00}(t)$ function of time only;

(ii) $\sum_{i,j=1}^{3} g^{ij}(x) \dfrac{\partial}{\partial t} g_{ij}(x)$ function of time only;

(iii) the connection 1-form Γ and Christoffel symbols C^k_{ij} satisfy

$$\sum_{i=1}^{3} g^{ij} [\Gamma_0, \Gamma_i] = 0, \qquad \sum_{i,j=1}^{3} g^{ij} \left[\Gamma_0, \frac{\partial \Gamma_j}{\partial x^i} + \Gamma_i \Gamma_j - \sum_{k=0}^{3} C^k_{ij} \Gamma_k\right] = 0,$$

$$\Gamma_0 = \Gamma_0(t) \text{ function of time only;}$$

(iv) the instantaneous Schrödinger operators $D_{\Sigma_t} = -\Delta + m^*(x)$ possess a complete system of time-independent eigenfunctions ζ_α (but with possibly time-dependent spectral resolution).

The mode solutions $T_\alpha(t)$ satisfy

$$\ddot{T}_\alpha(t) + F_\alpha(t)\dot{T}_\alpha(t) + G_\alpha(t)T_\alpha(t) = 0.$$

In what follows one comes across the following functional/harmonic analytical question (Q1): what is $\tilde{\mathcal{D}}(\mathcal{T})$, the Fourier image of $C_0^\infty(\mathcal{T})$? In fact, what we need is the following property (P1):

$$\tilde{f}(\alpha) \in \tilde{\mathcal{D}}(\mathcal{T}) \Rightarrow T_\alpha(t)\tilde{f}(\alpha) \in \tilde{\mathcal{D}}(\mathcal{T}), \quad \forall t.$$

Knowing the answer to (Q1) allows (P1) to be satisfied. However, (Q1) is not always easy to answer. As an alternative we have the following remedy.

Proposition 2. *If $\mathcal{T}$ is analytic and D_{Σ_t} has a strictly uniform spectrum, then (P1) can be satisfied regardless of the answer to (Q1).*

Now come to the weak solutions. By weak solutions we mean distributions which satisfy the field equation in the weak sense,

$$\mathcal{D}(\mathcal{T})'_0 = \{\varphi \in C_0^\infty(\mathcal{T})' : \quad D\varphi = 0\}.$$

Definition 2. *$\mathcal{D}(\mathcal{T})'_0$ is mode decomposed if a measure space $(\mathfrak{M}, d\mathfrak{m})$ and measurable families $\{T_\alpha(t) \in C^\infty(\mathbb{R})\}_{\alpha\in\mathfrak{M}}$, $\{X_\alpha(\vec{x}) \in C^\infty(\pi^{-1}(\Sigma))\}_{\alpha\in\mathfrak{M}}$ as above exist s.t. for any $\varphi \in \mathcal{D}(\mathcal{T})'_0$ there exist unique mode coefficients $a^\varphi, b^\varphi \in \tilde{\mathcal{D}}(\mathcal{T})'$ with*

$$\varphi(f) = a^\varphi(T_\alpha X_\alpha(f)) + b^\varphi(\bar{T}_\alpha X_\alpha(f)), \quad f \in C_0^\infty(\mathcal{T}).$$

Here

$$T_\alpha X_\alpha(f) = \int\limits_M d\mu_g(x)\langle T_\alpha(t)X_\alpha(\mathbf{x}), f(x)\rangle_{\mathfrak{g}}.$$

We extend the previous result to weak solutions as follows.

Proposition 3. *Weak solutions can be mode decomposed if and only if strong solutions can be decomposed and (P1) is satisfied.*

3. FOURIER/HARMONIC ANALYSIS

Definition 3. *An isometry of the field model described by $\mathcal{T} \xrightarrow{\pi} M$ is a bundle morphism $\Phi : \mathcal{T} \to \mathcal{T}$ covering an isometry of (M, g), preserving $\langle,\rangle_{\mathfrak{g}}$ and $D = \Box^\nabla + m^*(x)$.*

Classification of all cosmological models by isometry group G:

- $\dim G = 6$ FRW spacetimes, $O = SO(3)$;
- $\dim G = 4$ LRS spacetimes, $O = SO(2)$;
- $\dim G = 3$ Bianchi type spacetimes, $O = \{\mathbf{1}\}$.

In all cases besides Kantowski–Sachs model we have $G = \Sigma \rtimes O$ (semidirect homogeneous space), where $\Sigma = \mathrm{Bi}(N)/\Gamma$. Bi(I–IX) are the Bianchi I–IX groups, Γ — their discrete subgroups.

Definition 4. *Let $U_g : C^\infty(\mathcal{T}) \times G \mapsto C^\infty(\mathcal{T})$ be the left quasi-regular representation in the semidirect homogeneous bundle $\mathcal{T} \to G/O$. A bi-distribution $w \in (C_0^\infty(\mathcal{T}) \otimes C_0^\infty(\mathcal{T}))'$ is called invariant if*

$$(U_g \otimes U_g)w = w, \quad \forall g \in G.$$

In particular, quasifree homogeneous states of the field are given by such invariant bi-distributions.

Comparison of results

Lüders, Roberts, 1990	Here
Scalar field	Vector-valued field
FRW spacetimes	Semidirect spaces
Quasifree states	Bi-distributions
Additional continuity assumption	No assumptions

Proposition 4. *Any invariant bi-distribution w in a semidirect homogeneous bundle is given by*

$$w(f,h) = \mathrm{Tr}_{ij} u_w^{ij}((\bar{f}^i)^* * h^j), \quad f,h \in C_0^\infty(\mathcal{T})$$

with a distribution $u_w \in C_0^\infty(\mathcal{T} \otimes \mathcal{T})'$. Conversely, every such distribution u_w gives rise to an invariant bi-distribution w.

Here $f^*(x) = \bar{f}(x^{-1})$, and $f * h$ denotes the group convolution. The table gives a comparison of the scope of results available before in the literature [2] with what we have obtained. All these results with proofs and discussion can be found in [3].

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