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CHAOTIC SPIN PRECESSION IN ANISOTROPIC UNIVERSES AND FERMIONIC DARK MATTER

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We consider the precession of a Dirac particle spin in some anisotropic Bianchi universes. This effect is present already in the Bianchi-I universe. In the Bianchi-IX universe it acquires the chaotic character due to the stochasticity of the oscillatory approach to the cosmological singularity. The related helicity flip of fermions in the very early Universe may produce the sterile particles contributing to dark matter.

Мы рассматриваем прецессию спина дираковских частиц в некоторых анизотропных вселенных Бьянки. Этот эффект присутствует уже во вселенной Бьянки-I. Во вселенной Бьянки-IX он приобретает хаотический характер благодаря стохастичности осцилляторного приближения к космологической сингулярности. Связанный с этим переворот спиральности фермионов в очень ранней Вселенной может производить стерильные частицы, дающие вклад в темную материю.

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Dedicated to the memory of our teacher
Dmitry Vasilievich Shirkov.

INTRODUCTION

In almost all the applications of mathematical cosmology to the elaboration of observational data the isotropic Friedmann cosmological models are used. However, in the very early Universe, the effects of anisotropies could be essential. As is well known, the most simple and well-studied anisotropic cosmological models are the spatially homogeneous Bianchi models (see, e.g., [1,2]). Remarkably, already in the Bianchi models one can observe such an interesting and important phenomenon as the oscillatory approach to the cosmological singularity [3,4]. However, up to our knowledge, the behavior of quantum particles in the Bianchi universes was not studied in detail. We think that the filling of this gap can be of interest not only from the theoretical point of view, but can also reveal some interesting physical effects in the very early Universe.

Especially promising can be the study of motion of the Dirac particles (quarks and leptons) in gravitational fields. While this study has rather a long history [5], some essential progress was achieved in a recent series of papers [6–8]. In particular, the general expressions characterizing the spin motion in rather general gravitational fields were elaborated in paper [8].

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Here we apply this formalism to the study of the behavior of quantum particles with spin in some Bianchi universes. We found a novel effect of anisotropy induced spin precession revealed already in the simplest case of the Bianchi-I universe.

We perform also the comparison with the precession in the Bianchi-IX universe. First of all, two qualitatively different contributions to the angular velocity are present and, second, the oscillatory approach to the singularity [3] implies the stochasticity of the changes of the direction of the precession axis.

We also consider the possible physical consequences of these effects in very early Universe, including the appearance of effective magnetic field. The similar precession effects are also present for classical rotators due to the equivalence principle and might be manifested in the structure formation in very early Universe.

The equivalence principle implied also the helicity flip which is of special interest for the massive Dirac neutrinos. The neutrinos produced as active ones are becoming sterile due to gravity-induced helicity flip and may contribute to fermionic dark matter.

1. THE PRECESSION OF THE DIRAC PARTICLE IN A GRAVITATIONAL FIELD

Following paper [8], we present the general formula for the precession of the Dirac particle, adapted for the case of the Bianchi universes, where we shall use the sinchroneous frame. The metric can be presented as

$$ds^2 = dt^2 - \delta_{\hat{a}\hat{b}} W_c^{\hat{a}} W_d^{\hat{b}}\, dx^c\, dx^d, \tag{1}$$

where $a, b, \ldots$ are world spatial indices, while the ones with the hats are spatial tetrad indices. We shall also introduce the inverse matrix $W_{\hat{c}}^a$ such that $W_{\hat{c}}^a W_b^{\hat{c}} = \delta_b^a$.

In paper [8] it was shown that the average spin $\mathbf{s}$ in the semiclassical approximation is precessing with an angular velocity $\mathbf{\Omega}$ like

$$\frac{d\mathbf{s}}{dt} = \mathbf{\Omega} \times \mathbf{s} = (\mathbf{\Omega}_{(1)} + \mathbf{\Omega}_{(2)}) \times \mathbf{s}. \tag{2}$$

The velocities $\mathbf{\Omega}_{(1)}$ and $\mathbf{\Omega}_{(2)}$ correspond to gravitoelectric and to gravitomagnetic forces, respectively. Then,

$$\Omega_{(1)}^{\hat{a}} = \frac{1}{\varepsilon'} W_{\hat{c}}^d p_d \left(\frac{1}{2} \Upsilon \delta^{\hat{a}\hat{c}} - \varepsilon^{\hat{a}\hat{e}\hat{f}} C_{\hat{e}\hat{f}}^{\hat{c}} \right), \tag{3}$$

$$\Omega_{(2)}^{\hat{a}} = \frac{1}{2} \Xi^{\hat{a}} - \frac{1}{\varepsilon'(\varepsilon' + m)} \varepsilon^{\hat{a}\hat{b}\hat{c}} Q_{(\hat{b}\hat{d})} \delta^{\hat{d}\hat{n}} W_{\hat{n}}^k p_k W_{\hat{c}}^l p_l. \tag{4}$$

Here $C_{\hat{a}\hat{b}}^{\hat{c}}$ are anholonomity coefficients

$$C_{\hat{a}\hat{b}}^{\hat{c}} = W_{\hat{a}}^d W_{\hat{b}}^e \partial_{[d} \dot{W}_{e]}^{\hat{c}}, \quad C_{\hat{a}\hat{b}\hat{c}} = g_{\hat{c}\hat{d}} C_{\hat{a}\hat{b}}^{\hat{d}}. \tag{5}$$

Then,

$$Q_{\hat{a}\hat{b}} = g_{\hat{a}\hat{c}} W_{\hat{b}}^d \dot{W}_d^{\hat{c}}, \tag{6}$$

$$\Upsilon = -\varepsilon^{\hat{a}\hat{b}\hat{c}} C_{\hat{a}\hat{b}\hat{c}}, \tag{7}$$

$$\Xi_{\hat{a}} = \varepsilon_{\hat{a}\hat{b}\hat{c}} Q_{\hat{b}\hat{c}}. \tag{8}$$

The motion of the particle is characterized by its momentum p_a and by the energy $\varepsilon' = \sqrt{m^2 + \delta^{\hat{c}\hat{d}} W^a_{\hat{c}} W^b_{\hat{d}} p_a p_b}$. It can be absorbed together with the particle mass m and its momentum p_a by introducing the velocity v_a. Thus, the precession velocities are

$$\Omega^{\hat{a}}_{(1)} = v_{\hat{c}} \left(\frac{1}{2} \Upsilon \delta^{\hat{a}\hat{c}} - \varepsilon^{\hat{a}\hat{e}\hat{f}} C^{\hat{c}}_{\hat{e}\hat{f}} \right), \tag{9}$$

$$\Omega^{\hat{a}}_{(2)} = \frac{1}{2} \Xi^{\hat{a}} - \frac{\gamma}{\gamma + 1} \varepsilon^{\hat{a}\hat{b}\hat{c}} Q_{(\hat{b}\hat{d})} \delta^{\hat{d}\hat{n}} v_{\hat{n}} v_{\hat{c}}, \tag{10}$$

where $\gamma = 1/\sqrt{1 - v^2}$ is a Lorentz factor.

2. THE SPINNING PARTICLE IN THE BIANCHI-I UNIVERSE

The simplest spatially homogeneous and anisotropic universe is that of Bianchi-I type, whose metric is [1]:

$$ds^2 = dt^2 - a^2(t)(dx^1)^2 - b^2(t)(dx^2)^2 - c^2(t)(dx^3)^2. \tag{11}$$

Comparing this expression with Eq. (1), we have the following expressions for the nonvanishing elements of the matrix $W^b_{\hat{a}}$:

$$W^{\hat{1}}_1 = a(t), \quad W^{\hat{2}}_2 = b(t), \quad W^{\hat{3}}_3 = c(t). \tag{12}$$

The elements of the inverse matrix are

$$W^1_{\hat{1}} = \frac{1}{a(t)}, \quad W^2_{\hat{2}} = \frac{1}{b(t)}, \quad W^3_{\hat{3}} = \frac{1}{c(t)}. \tag{13}$$

As is well known, the anholonomity coefficients for the Bianchi-I model are equal to zero. Hence, $\Upsilon = 0$ too and the "gravitoelectric" contribution $\boldsymbol{\Omega}_{(1)}$ disappears. Then, the nonvanishing coefficients of the matrix $Q_{\hat{a}\hat{b}}$ are

$$Q_{\hat{1}\hat{1}} = -\frac{\dot{a}}{a}, \quad Q_{\hat{2}\hat{2}} = -\frac{\dot{b}}{b}, \quad Q_{\hat{3}\hat{3}} = -\frac{\dot{c}}{c}. \tag{14}$$

Correspondingly, also the vector $\Xi_{\hat{a}}$ disappears. Finally, the nonvanishing components of the "gravitomagnetic" contribution to the precession of the Dirac particle in the Bianchi-I universe are, up to cyclic permutations,

$$\Omega^{\hat{1}}_{(2)} = \frac{\gamma}{\gamma + 1} v_{\hat{2}} v_{\hat{3}} \left(\frac{\dot{b}}{b} - \frac{\dot{c}}{c} \right). \tag{15}$$

The solution of the Einstein equations for the empty Bianchi-I universe — the Kasner solution [9, 10] — is

$$a(t) = a_0 t^{p_1}, \quad b(t) = b_0 t^{p_2}, \quad c(t) = c_0 t^{p_3}, \tag{16}$$

where the Kasner indices p_1, p_2, and p_3 satisfy the relations

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1. \tag{17}$$

Correspondingly, Eq. (15) becomes

$$\Omega^{\hat{1}}_{(2)} = \frac{\gamma}{\gamma+1} v_{\hat{2}} v_{\hat{3}} \left(\frac{p_2 - p_3}{t} \right) \tag{18}$$

and has some similarity to the Euler equations for rigid body rotation with p_i being correspondent with the moments of inertia.

Obviously, this effect can be essential in the early Universe, i.e., at the very small values of the proper cosmic time t.

Besides the Kasner solution for an empty Bianchi-I universe, some other general exact solutions for the universes with this type of symmetry exist. First of all, it is the Heckmann–Schucking solution for the Bianchi-I universe, filled with dust [11], which can be also generalized for the case when not only dust, but also a cosmological constant and stiff matter are present [12]. The Heckmann–Schucking solution can be written down in the following form:

$$\begin{aligned} a(t) &= a_0 t^{p_1} (t_0 + t)^{\frac{2}{3} - p_1}, \\ b(t) &= b_0 t^{p_2} (t_0 + t)^{\frac{2}{3} - p_2}, \\ c(t) &= c_0 t^{p_3} (t_0 + t)^{\frac{2}{3} - p_3}. \end{aligned} \tag{19}$$

Here, the constant t_0 depends on the quantity of the dust-like matter in the universe. It grows when the quantity of dust diminishes. It is easy to see that for the small values of the cosmic time parameter $t \ll t_0$, the solution (19) practically coincides with the Kasner solution (16), (17). In the opposite case, when $t \gg t_0$, all the scale factors $a(t), b(t)$, and $c(t)$ are proportional to $t^{2/3}$; i.e., the universe behaves as the flat Friedmann universe filled with dust. Thus, we observe the isotropization of the universe. Let us substitute the factors (19) into the formula (15) obtaining, up to cyclic permutations,

$$\Omega^{\hat{1}}_{(2)} = \frac{\gamma}{\gamma+1} v_{\hat{2}} v_{\hat{3}} \frac{(p_2 - p_3) t_0}{t(t_0 + t)}. \tag{20}$$

At the large values of $t \gg t_0$ we have

$$\Omega^{\hat{1}}_{(2)} = \frac{\gamma}{\gamma+1} v_{\hat{2}} v_{\hat{3}} \frac{(p_2 - p_3) t_0}{t^2} \left(1 + o\left(\frac{t_0}{t} \right) \right). \tag{21}$$

Thus, we see that the isotropization implies a very rapid decreasing of the precession effect.

3. PRECESSION IN THE BIANCHI-IX UNIVERSE

The matrix $W^{\hat{b}}_{\ a}$ for the Bianchi-IX metric (see, e.g., [2]) can be written as

$$W^{\hat{b}}_{\ a} = \begin{pmatrix} -a \sin x^3 & a \sin x^1 \cos x^3 & 0 \\ b \cos x^3 & b \sin x^1 \sin x^3 & 0 \\ 0 & c \cos x^1 & c \end{pmatrix}, \tag{22}$$

where a, b, and c are some functions of time as usual. Its inverse matrix $W^{c}_{\hat{b}}$ is

$$W^{c}_{\hat{b}} = \begin{pmatrix} -\frac{1}{a}\sin x^3 & \frac{1}{b}\cos x^3 & 0 \\ \frac{1}{a}\frac{\cos x^3}{\sin x^1} & \frac{1}{b}\frac{\sin x^3}{\sin x^1} & 0 \\ -\frac{1}{a}\frac{\cos x^1 \cos x^3}{\sin x^1} & -\frac{1}{b}\frac{\sin x^3 \cos x^1}{\sin x^1} & \frac{1}{c} \end{pmatrix}. \tag{23}$$

The nonvanishing anholonomity coefficients are

$$C^{\hat{3}}_{\hat{1}\hat{2}} = \frac{c}{ab}, \text{ and cyclic permutations.} \tag{24}$$

Then,

$$\Upsilon = 2\left(\frac{c}{ab} + \frac{b}{ac} + \frac{a}{bc}\right). \tag{25}$$

The nonvanishing coefficients of the matrix $Q_{\hat{a}\hat{b}}$ are the same as in Eq. (14). Hence, $\Xi_{\hat{a}}$ is again equal to zero. The "gravitoelectric" precession velocity, up to cyclic permutations, is

$$\Omega^{\hat{1}}_{(1)} = v^{\hat{1}}\left(\frac{c}{ab} + \frac{b}{ac} - \frac{a}{bc}\right). \tag{26}$$

The expressions for the components of the "gravitomagnetic" velocity $\mathbf{\Omega}_{(2)}$ remain the same as in the Bianchi-I universe (see, Eq. (15)).

Thus, we have seen that the "gravitomagnetic" velocity in the Bianchi-IX universe is the same as in the Bianchi-I universe; however, in the Bianchi-IX universe there is also the "gravitoelectric" precession. The presence of this term (26) is connected with the presence of a spatial curvature in the Bianchi-IX universe, in contrast to the Bianchi-I universe. It is connected with the fact that the anholonomity coefficients are nonvanishing in the Bianchi-IX universe.

Now, let us discuss what happens with the precession of the Dirac particles in the Bianchi-IX universe, evolving towards the cosmological singularity. As was discovered at the end of the sixties, the Bianchi-IX universe approaches the singularity in an oscillating way [3,4], and these oscillations have chaotic character [13]. The evolution towards the singularity can be described by the subsequence of the periods when the universe behaves like a Kasner universe (16), (17), separated by time intervals when one Kasner regime is substituted by another one. Let us remind how these changes occur. The Kasner indices p_1, p_2, and p_3 can be expressed through the Lifshitz–Khalatnikov parameter u [10] as

$$p_1 = -\frac{u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}, \tag{27}$$

where $u > 1$. The perturbative terms in the Einstein equations, connected with the spatial curvature, induce the transition to another Kasner regime (which is called "epoch" [1, 3]). Such that

$$p'_1 = p_2(u-1), \quad p'_2 = p_1(u-1), \quad p'_3 = p_3(u-1). \tag{28}$$

It means that if during the preceding epoch, the universe is expanding along the first axis and contracting along the second and the third axes, in the successive epoch it begins expanding along the second axis; i.e., the first and the second axes change their roles. There is another type of transition when the parameter u becomes less than 1. In this case, the change of the "Kasner era" occurs [1, 3]. This change is described by the following formula:

$$p_1' = p_1\left(\frac{1}{u}\right), \quad p_2' = p_3\left(\frac{1}{u}\right), \quad p_3' = p_2\left(\frac{1}{u}\right). \tag{29}$$

The transition from one Kasner era to another can be described by the mapping transformation of the interval $[0,1]$ into itself by the formula

$$Tx = \left\{\frac{1}{x}\right\}, \quad x_{s+1} = \left\{\frac{1}{x_s}\right\}, \tag{30}$$

where curly brackets stand for the fractional part of a number. This transformation belongs to the so-called expanding transformations of the interval $[0,1]$, i.e., transformations $x \sim f(x)$ with $|f'(x)| > 1$. Such transformations possess the property of exponential instability: if we take initially two close points, their mutual distance increases exponentially under the iterations of the transformations. It is well known that the exponential instability leads to the appearance of strong stochastic properties [13].

Now, let us describe what happens with our angular velocities $\mathbf{\Omega}_{(1)}$ and $\mathbf{\Omega}_{(2)}$ when the universe oscillating approaches the singularity. The expression for $\mathbf{\Omega}_{(2)}$ can be written as

$$\begin{aligned} \Omega_{(2)}^{\hat{1}} &= \frac{\gamma}{(\gamma+1)t} v_{\hat{2}} v_{\hat{3}} \frac{1-u^2}{1+u+u^2}, \\ \Omega_{(2)}^{\hat{2}} &= \frac{\gamma}{(\gamma+1)t} v_{\hat{1}} v_{\hat{3}} \frac{2u+u^2}{1+u+u^2}, \\ \Omega_{(2)}^{\hat{3}} &= -\frac{\gamma}{(\gamma+1)t} v_{\hat{1}} v_{\hat{2}} \frac{1+2u}{1+u+u^2}. \end{aligned} \tag{31}$$

It is easy to show that after the change of the Kasner epoch, the new expressions for the components of the velocity can be obtained by substitution $u \to -u$ in this equation. It means that the first component does not change the sign, the third component changes the sign, while the second component changes the sign if $u > 2$.

After the change of the Kasner era, all the components of the velocity $\mathbf{\Omega}_{(2)}$ just change the sign, preserving the absolute values, as it follows immediately from (29).

The leading terms for the components of the velocity $\mathbf{\Omega}_{(1)}$ are

$$\begin{aligned} \Omega_{(1)}^{\hat{1}} &\sim -v^{\hat{1}}(t)^{\left(-1-\frac{2u}{1+u+u^2}\right)}, \\ \Omega_{(1)}^{\hat{b}} &\sim v^{\hat{b}}(t)^{\left(-1-\frac{2u}{1+u+u^2}\right)}, \quad b = 2, 3. \end{aligned} \tag{32}$$

The change of epochs boils down to

$$\begin{aligned} \Omega_{(1)}^{\hat{2}} &\sim -v^{\hat{2}}(t)^{\left(-1-\frac{2u-2}{1-u+u^2}\right)}, \\ \Omega_{(1)}^{\hat{a}} &\sim v^{\hat{a}}(t)^{\left(-1-\frac{2u-2}{1-u+u^2}\right)}, \quad a = 1, 3. \end{aligned} \tag{33}$$

Curiously, the change of eras leaves leading terms under consideration intact. Thus, we have seen that the precession of the Dirac particle in the Bianchi-IX universe evolving towards the singularity also follows a chaotic pattern.

4. DISCUSSION AND OUTLOOK

We have seen that the precession of a Dirac particle spin exists already in the Bianchi-I universe. Interestingly, the Kasner indices play the role similar to the moments of inertia in the Euler equation for the rigid body precession. In the Bianchi-IX universe the precession acquires the chaotic character due to the stochasticity of the oscillatory approach to the cosmological singularity [3, 4]. Remarkably, the formulae for the changes of the precession direction are nicely expressible in terms of the Lifshitz–Khalatnikov parameter u.

What physical consequences could it have for the very early Universe?

Let us note first that precession due to anisotropy of the Universe may be considered as generated by some effective magnetic field. The latter may be easily obtained by equating the angular velocity to that of the Larmor precession. For definiteness, in the Bianchi-I universe it reads, up to cyclic permutations,

$$H^{\hat{1}} = \frac{m\gamma}{2eg(\gamma+1)} v_{\hat{2}} v_{\hat{3}} \left(\frac{p_2 - p_3}{t} \right). \tag{34}$$

As a result, the anisotropy of the Universe provides all the Dirac particles with effective anomalous magnetic moments. In particular, the transitions between the Dirac neutrinos and their sterile partners may be induced in such a way. Moreover, due to equivalence principle, these conclusions may be extended to particles of any spin [14] and also to classical rotators [7]. The latter fact opens the possibility to study the role of the discussed precession effects for the formation of structures in the very early Universe and angular momentum of cosmic strings [15].

Also, equivalence principle leads to the precession frequencies of spin and velocity differing by a factor of 2, so that the helicity is conserved in the noninertial frame rotating with the same frequency, but it is flipped [14] in the inertial frame. This effect is especially interesting for the massive Dirac neutrinos. If they are produced in the very early Universe as active ones, the gravity-induced helicity flip may turn them to sterile neutrinos which remain in this state after the Universe becomes isotropic and contribute to fermionic dark matter. As soon as the rotation period is defined by the age of the Universe in the anisotropic phase, the amounts of sterile and active neutrinos at the end of this phase are, generally speaking, of the same order:

$$\frac{N_{\text{sterile}}}{N_{\text{active}}} \sim 1. \tag{35}$$

If the spin happens to perform the rotation for an angle close to π, the velocity will rotate for the angle close to 2π, and most of the fermions will become sterile:

$$\frac{N_{\text{sterile}}}{N_{\text{active}}} \gg 1. \tag{36}$$

This opens, in principle, the possibility to attribute the dark matter to the contribution of light sterile neutrinos, whose abundance would be much larger than that of thermal ones. The validity of (36) would require a sort of fine tuning, but not too strong one, as the required excess of sterile neutrinos is about two orders of magnitude, and the closeness of the rotation angle to π should be also at the percent level.

The anisotropic metrics, in the case of some scale parameters being much smaller than others, may provide the model of transitions between spaces of different (effective) dimension [16–18]. The spin dynamics in that case is manifesting the interesting effects [19]. Other interesting directions of investigation could be connected with the study of the spin precession in the Bianchi-II universes, in the generalized Melvin cosmologies in the presence of electromagnetic fields [20], and in the double Kasner universes [15]. We hope to study these topics in detail in future publications.

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