Письма в ЭЧАЯ. 2007. Т. 4, № 2(138). С. 295–299

GATES FOR QUANTUM COMPUTING INDUCED FROM MONODROMY OPERATORS

G. Giorgadze[1]

Institute of Cybernetics of the Georgian Academy of Sciences, Tbilisi, Georgia

The monodromy approach to quantum computing is based on a holomorphic vector bundle with meromorphic connection. The gates for computation are obtained from the monodromy matrices of the connection.

В работе рассматривается модель квантовых вычислений на основе голоморфного векторного расслоения с мероморфной связностью. Квантовые вентили строятся из матриц монодромии данной связности.

1. CONSTRUCTION OF THE UNIVERSAL SET OF GATES BY FUCHSIAN SYSTEM

In this section we present the ideology for constructing universal set of gates for quantum computation using Fuchsian system of differential equations [1]. This system has quantum mechanical sense; besides, under certain conditions on the Hamiltonian of quantum system the corresponding Schrödinger equation reduces to Fuchsian system.

Unlike holonomic quantum computation [2], in the monodromic approach to quantum computation [3], connection of the vector bundle, which is the main ingredient of this model, is completely integrable. From the complete integrability of connection it follows that the holonomy representation of the loop space reduces to the monodromy representation of fundamental group of base of the bundle. From one-dimensionality of the base of holomorphic bundle automatically follows integrability condition of the connection. For this reason we begin by considering this case.

Denote by $X_m = CP^1 \setminus \{s_1, \ldots, s_m\}$ and consider holomorphic vector bundle $E \to X_m$ with connection ω with logarithmic singular points $s_1, \ldots, s_m$. It means that ω has the form

$$\omega = \sum_{j=1}^{m} \frac{A_j}{z - s_j} dz,$$

where A_j, $j = 1, \ldots, m$ are $N \times N$ constant matrices. Let $\gamma_1, \ldots, \gamma_m$ be generators of fundamental group $\pi_1(X_m, z_0)$, where $z_0 \in X_m$, of the complex manifold X_m.

[1]E-mail: gg@cybernet.ge

The Chen iterated integral gives the monodromy representation

$$\rho : \pi_1(CP^1 \setminus \{s_1, \ldots, s_m\}, z_0) \to GL_N(C), \gamma_i \mapsto M_j \tag{1}$$

by

$$M_j = 1 + \int\limits_{\gamma_j} \omega + \int\limits_{\gamma_j} \omega\omega + \ldots$$

The fundamental group $\pi_1(X_m, z_0)$ acts on the space of holomorphic sections of the bundle $E \to X_m$ as $v = M_j u$, where $u, v \in C^N$.

The space of holomorphic sections of the bundle $E \to X_m$ isomorphic to zero cohomology group $H^0(X_m, \mathcal{O}(E))$, where $\mathcal{O}(E)$ denotes the sheaf of the holomorphic sections of the bundle. On the other hand, $H^0(X_m, \mathcal{O}(E))$ is isomorphic to space of solutions of the system of differential equations

$$df = \omega f. \tag{2}$$

In the monodromic approach to quantum computing, one is interested in the inverse procedure.

Let $U_1, \ldots, U_k$ be the set of the unitary operators which are necessary for some quantum algorithm. Here is a natural question: does there exist a system (2) with monodromy (1) which gives the gates $U_1, \ldots, U_k$ as the monodromy? The answer is given by the following result.

There exists a location of points $s_1, \ldots, s_k$ on the Riemann sphere CP^1 such that, for given matrices $U_1, \ldots, U_k$, there exists a meromorphic connection ω which has at points $s_1, \ldots, s_k$ the poles of first order and monodromy representation induced from ω coincides with the given set of unitary matrices. Moreover, the points $s_1, \ldots, s_k$ can be chosen to lie on the real axis.

$\lhd$ The proof follows from solvability of Hilbert's 21st problem for Fuchsian system on the Riemann sphere [1]. More precisely, it is known that for fixed data $\{s_1, \ldots, s_k\} \in CP^1$, the set of collections $\{U_1, \ldots, U_k\} \in SU(N)$ for which Hilbert's 21st problem is solvable, is a dense set in the $CP^1 \times (SU(N))^k$. $\rhd$

Is is appropriate to remark that certain Fuchsian systems are Shrödinger-type equation [4], which enables one to use the theory of Fuchsian equations in quantum computing. It is also known [5] that the universal set of gates are given by all 2×2-unitary operators and one unitary entangled operator $R : C^2 \otimes C^2 \to C^2 \otimes C^2$. It means that there is a vector $|uv\rangle = |u\rangle \otimes |v\rangle \in C^2 \otimes C^2$ such that $R|uv\rangle$ is not decomposable as a tensor product of two qubits.

Below we give the way to obtain all 2×2-unitary operators from the Fuchsian-type system (2). It is known that a compact Lie group has two generators. Let for $SU(2)$ these generators be g_1, g_2. Choose three points $s_1, s_2, s_3 \in C$ and generators $\gamma_1, \gamma_2, \gamma_3$ of $CP^1 \setminus \{s_1, s_2, s_3\}$. It is also known that $\gamma_1, \gamma_2, \gamma_3$ satisfy the condition $\gamma_1\gamma_2\gamma_3 = 1$. Consider the representation of $\rho : \pi_1(CP^1 \setminus \{s_1, s_2 s_3\}) \to SU(2)$, defined by relation $\rho(\gamma_j) = g_j$, $j = 1, 2, 3$. For the Riemann data $((s_1, s_2, s_3), (g_1, g_2, g_3))$, Hilbert's 21st problem is solvable; therefore, there exist 2×2-matrices A_1, A_2, A_3, such that the monodromy representation of the Fuchsian system

$$df = \left(\frac{A_1}{z - s_1} + \frac{A_2}{z - s_2} + \frac{A_n}{z - s_3} \right) f$$

coincides with ρ.

Let $E_\rho \to CP^1$ be the holomorphic vector bundle induced from ρ. The one-form

$$\omega = \frac{A_1}{z - s_1} + \frac{A_2}{z - s_2} + \frac{A_n}{z - s_n}$$

defines a connection of this bundle and if we interpret the fibre C^2 as the space of qubits, then the monodromy matrices acting on the qubits play the role of quantum gates.

The most natural way from a two-dimensional vector bundle with holomorphic connection to the unitary operator from $C^2 \otimes C^2$ to $C^2 \otimes C^2$ is to consider two bundles with connections (E_1, ω_1) and (E_2, ω_2) and build the rank-4 holomorphic vector bundle $E_1 \otimes E_2 \to CP^1$ with fibre $C^2 \otimes C^2$ and with connection $\omega_1 \otimes 1 + 1 \otimes \omega_2$. However, the monodromy matrices of this connection do not give an entangled operator. To obtain an entangled operator from Fuchsian system $df = \omega f$, it is necessary to take $\omega = \dfrac{A}{z} dz$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}.$$

The monodromy matrix $M = \mathrm{e}^{iA}$ of this system is an entangled operator. The check of this fact is possible by online programme www.physics.uq.edu.au/gqc. This programme also enables one to explicitly construct CNOT operator from unitary 2×2 matrices and from

$$\begin{pmatrix} 0.6i & 0.4i & -0.6 - 0.2i & 0.4i \\ 0.4i & -0.6i & 0.4i & -0.6 + 0.2i \\ -0.6 - 0.2i & 0.4i & 0.6i & 0.4i \\ 0.4i & -0.6 + 0.2i & 0.4i & -0.6i \end{pmatrix},$$

which is similar to M.

2. MULTIDIMENSIONAL SYSTEMS AND MODELS

The model of quantum computation introduced in the first section may be called one-dimensional since the base of the bundle is a one-dimensional complex manifold. In this section we give a multidimensional generalization of the constructions given above. Here we remark that the multidimensionality of base immediately gives rise to the problem of integrability of the system type (2), where the function f is now a function depending on the vector $z = (z_1, \ldots, z_p)$ with values in a vector space or in a Lie algebra.

An example of such a generalization is the Knizhnik–Zamolodchikov equation in conformal field theory. These system of equations is a differential equation for n-point correlation function $\psi(z_1, \ldots, z_n)$ of conformal field theory for Kac–Moody algebra $\widehat{\mathcal{G}}$. The points $z_1, \ldots, z_n$ are distinct points on the complex line, and the correlation function takes values in an n-fold tensor product $V_1 \otimes \ldots \otimes V_n$ of representations of a finite-dimensional simple Lie algebra $\mathcal{G}$. The system of equations has the form

$$(c + h^\vee) \frac{d\psi}{dz_i} = \left(\sum_{j=1, j \neq i}^{n} \frac{\Omega_{ij}}{z_i - z_j} \right) \psi, \quad i = 1, \ldots, n, \tag{3}$$

where Ω is the symmetric Casimir tensor $\Omega = \sum x_i \otimes x^i \in \mathcal{G} \otimes \mathcal{G}$ corresponding to the invariant scalar product on $\mathcal{G}$ and Ω_{ij} denotes the action of Ω on the ith and jth slot of the n-fold tensor ψ. The number $h^\vee$ is the dual Coxeter number of $\widehat{\mathcal{G}}$ and the complex number c is called the central charge.

The Knizhnik–Zamolodchikov equation defines a connection $\omega = \sum \Omega_{ij} d\log(z_i - z_j)$ with logarithmic singularity along the divisor $D = \bigcup_{i=1}^{n} \{z_i - z_j = 0\}$ of the vector bundle on the $X_n = C^n \setminus D$ with fibre $V_1 \otimes \ldots \otimes V_n$. The system (3) is integrable and is invariant for the $\mathcal{G}$-action on ψ. Hence, it defines a monodromy representation of the fundamental group $B_n = \pi_1(C^n \setminus D, z_0)$. This monodromy representation takes values in the space of $\mathcal{G}$-intertwiners between tensor products of $\mathcal{G}$-modules (see [6]).

In [7] the integrable models of quantum mechanics are classified which are invariant under the action of a Weyl group with certain assumptions. For the case of B_N, $N \geqslant 3$ the generic model of the family of integrable quantum systems, which is called the Calogero–Moser–Sutherland system, coincides with the BC_N Inozemtsev model. The eigenvalue problem for the Hamiltonian of the BC_1 Inozemtsev model is transformed to the Heun equation with full parameters [8]. By definition the Heun equation is a second-order Fuchsian system differential equation with four regular singular points. More precisely, the Hamiltonian of the BC_1 Inozemtsev model is given by

$$H := -\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i), \tag{4}$$

where $\wp(x)$ is the Weierstrass $\wp$ function with periods $(2\omega_1, 2\omega_3)$, $\omega_0 = 0$, $\omega_2 = -(\omega_1+\omega_3)$, and l_i $(i = 0, 1, 2, 3)$ are coupling constants. Let $f(x)$ be an eigenfunction of H with eigenvalue E, i.e.,

$$(H-E)f(x) = \left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i+1)\wp(x+\omega_i) - E\right) f(x) = 0. \tag{5}$$

This equation can be transformed to the Heun equation (see [8]):

$$\left(\left(\frac{d}{dw}\right)^2 + \left(\frac{\gamma}{w} + \frac{\delta}{w-1} + \frac{\epsilon}{w-t}\right)\frac{d}{dw} + \frac{\alpha\beta w - q}{w(w-1)(w-t)}\right) g(w) = 0,$$

with the condition $\alpha + \beta + 1 = \gamma + \delta + \epsilon$.

Conversely, if a Fuchsian differential equation with four regular singularities is given, we can transform it into equation (5) with suitable values of the exponents of the singular points. Thus, the relationship between the BC_1 Inozemtsev model and the Heun equation is very explicit.

It is known that system of quantum particles without reflection described by scattering matrix $S(u)$, where u is the angle between the trajectories of particles, satisfies the Yang–Baxter type equation

$$S(u)S(u+v)S(v) = S(v)S(u+v)S(u).$$

The symmetry of that equation is characterized by the permutation group S_n. If in addition we take into account reflection, then we obtain one equation in addition. If we denote by

$K(v)$ the reflection matrix of particles, then the equation have the form

$$K(u)S(2u+v)K(u+v)S(v) = S(v)K(u+v)S(2u+v)K(u).$$

The symmetry of such a system is connected Lie algebras associated with root systems. A physically meaningful example of a system with reflection is given by a Gaudin magnetic. The description of Gaudin magnetic is related to solution of the above Knizhnik–Zamolodchikov equation which is associated with a B_N-type root system. The formalism for the construction of quantum gates described above is applicable for arbitrary root systems.

REFERENCES

1. *Anosov D. V., Bolibruch A. A.* The Riemann–Hilbert Problem, Aspects of Mathematics. Vieweg; Braunschweig; Wiesbaden, 1994.

2. *Zanardi P., Rasetti M.* Holonomic Quantum Computation // Phys. Lett. A. 1999. V. 264. P. 94.

3. *Giorgadze G.* Monodromic Approach to Quantum Computing // Intern. J. Mod. Phys. B. 2002. V. 16, No. 30. P. 4593–4605.

4. *Giorgadze G., Khimshiashvili G.* On Schrödinger Equations of Okubo Type // J. Dyn. Cont. Syst. 2004. V. 10, No. 2. P. 171–186.

5. *Brylinski J. L., Brylinski R.* Universal Quantum Gates // Mathematics of Quantum Computation. Boca Raton, Florida: Chapman & Hall/CRC Press, 2002.

6. *Giorgadze G.* Regular Systems on Riemann Surfaces // J. Math. Sci. 2003. V. 118(5). P. 5347–5399.

7. *Oshima T., Sekiguchi H.* Commuting Families of Differential Operators Invariant under the Action of a Weyl Group // J. Math. Sci. Univ. Tokyo. 1995. V. 2. P. 1–75.

8. *Takemura K.* The Heun Equation and the Calogero–Moser–Sutherland System I: The Bethe Ansatz Method // Commun. Math. Phys. 2003. V. 235. P. 467–494.