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КОМПЬЮТЕРНЫЕ ТЕХНОЛОГИИ В ФИЗИКЕ

ASYMPTOTIC HAMILTONIAN REDUCTION FOR THE DYNAMICS OF A PARTICLE ON A SURFACE

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We consider the motion of a particle on the surface generated by a small perturbation of the standard sphere. The key observation is that a trajectory of the particle has the shape of a coil, and one may qualitatively describe the turns of the latter as a precessing great circle of the sphere. Thus, we change the configuration space of the initial problem for the space of great circles on the sphere. The construction enables us to derive a subsidiary Hamiltonian system having the shape of equations for the top with a fourth-order Hamiltonian. The subsidiary system provides the detailed asymptotic description of the particle's motion in terms of graphs on the standard sphere.

Рассматривается движение частицы на поверхности, полученной путем малых возмущений стандартной сферы. Ключевым фактом является то, что траектория частицы имеет форму кольца, и существует возможность качественного описания его поворотов как прецессии большого «круга» сферы. Таким образом, мы изменяем конфигурационное пространство исходной проблемы для пространства больших кругов на сфере. Конструкция позволяет получить вспомогательную гамильтонову систему, имеющую форму уравнений для верхней поверхности с гамильтонианом 4-го порядка. Вспомогательная система позволяет асимптотически точно описывать движение частиц в терминах графов на стандартной сфере.

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INTRODUCTION

The dynamics of a particle which is allowed to move on a smooth surface and not acted upon by any forces, is the classical problem in analytical dynamics [1]. The orbits of the particle (or geodesics on the surface) are generally hard to find. Even in the specific case of ellipsoid there is a need for the use of the analytical mechanics [2]. Nonetheless, the latter runs across serious difficulties in studying geodesics. The reason for this lies in that there is generally only the energy conservation; all other integrals of motion, even the angular momentum, being absent owing to an asymmetry of the problem.

To overcome these difficulties we shall employ the construction of a subsidiary Hamiltonian problem that is easy to solve and provides a qualitative description for the ensemble of orbits. We shall restrict our problem and suppose the surface to be a perturbed standard sphere. Thus, we may use the perturbation theory for studying the orbits, and construct a subsidiary Hamiltonian system which acts in a phase space different from the initial one, its points being great circles of the standard sphere. This approach enables us to give a fairly detailed picture of the ensemble of orbits by means of graphs on the standard sphere; the vertices of the graphs corresponding to orbits which are asymptotically closed and the edges of the graphs to orbits joining the almost closed ones.

To be specific, the central idea relies on the circumstance that if the surface does not differ substantially from a sphere, the great circles of the latter may serve a good approximation to the surface's geodesics, if they are short enough, and one can visualize them as winding up in coils; loops or rings of the coil corresponding to great circles of the sphere (see Fig. 1). Hence approximating the successive rings by great circles, we may describe the change in the position of the rings by the motion of a great circle, which in its turn is determined by the normal vector $\mathbf{L}$ of the plane cutting the sphere along the great circle. To cast this picture in a more quantitative form, we may use the fact that the normal vector $\mathbf{L}$ is the angular momentum of the particle moving along the great circle.

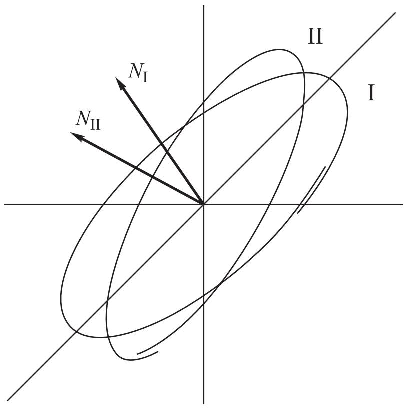

Fig. 1. Two coils I, II of an orbit on the surface $\varphi(\mathbf{x}) = 0$; vectors $N_{\rm I}$ and $N_{\rm II}$ are the normals to the planes of great circles approximating the coils

Averaged Equations of Geodesics. The equations determining geodesics on a surface given by the equation $\varphi(\mathbf{x}) = 0$ can be cast in the form of the equation [1]

$$\ddot{\mathbf{x}} = \lambda \frac{\partial \varphi}{\partial \mathbf{x}}. \tag{1}$$

The Lagrangian multiplier can be found explicitly, so that the equation of motion, in the form that does not involve λ, reads

$$\ddot{\mathbf{x}} = -\frac{\dot{\mathbf{x}} \dfrac{\partial^2 \varphi}{\partial \mathbf{x}^2} \dot{\mathbf{x}}}{\left(\dfrac{\partial \varphi}{\partial \mathbf{x}}\right)^2} \frac{\partial \varphi}{\partial \mathbf{x}}. \tag{2}$$

In this paper we consider surfaces that do not differ substantially from sphere, and assume that their equations be of the form

$$\varphi(\mathbf{x}) = \sum_{i=1}^{3} (x_i^2 + \varepsilon_i x_i^4) - 1 = 0, \tag{3}$$

where ε_i are small.

Consider the angular momentum $\mathbf{L} = \mathbf{x} \times \dot{\mathbf{x}}$. The equations for its components can be derived from (2) and (3). Then we use the method of averaging. Generally, the approach relies on studying the evolution equations for integrals of motion of the unperturbed system, i.e., in our case the normals to the planes of the great circles, with respect to the basic periodic solution of the latter. The averaging serves as a filter separating the main regular part of the solution from the oscillating one caused by small terms considered as perturbation, see [3].

We may write the basic equation for the particle's motion on the sphere of unit radius in the form

$$\mathbf{x} = \cos{(\omega t + \theta)} \mathbf{e}_1 + \cos{(\omega t + \theta)} \mathbf{e}_2,$$

vector $\mathbf{e}_3$ of the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ being parallel to $\mathbf{L}$. The angular velocity ω is given by the equation $\omega^2 = \dot{\mathbf{x}}^2 = L^2$, valid to within the first order of perturbation. With the help of the equations given above and neglecting terms of the second, and higher, order in the ε_i, we can derive equations for $\mathbf{L}$ that do not involve $\mathbf{x}$. After averaging these equations read

$$\begin{aligned}
\dot{L_1} &= \frac{3}{4}\,\frac{L_2L_3}{L^2}\left[(\varepsilon_3-\varepsilon_2)L_1^2+\varepsilon_3L_2^2-\varepsilon_2L_3^2\right],\\
\dot{L_2} &= \frac{3}{4}\,\frac{L_3L_1}{L^2}\left[-\varepsilon_3L_1^2+(\varepsilon_1-\varepsilon_3)L_2^2+\varepsilon_1L_3^2\right],\\
\dot{L_3} &= \frac{3}{4}\,\frac{L_1L_2}{L^2}\left[\varepsilon_2L_1^2-\varepsilon_1L_2^2+(\varepsilon_2-\varepsilon_1)L_3^2\right].
\end{aligned}\tag{4}$$

It is worth noting that equations (4) have the Hamiltonian form determined by the usual Poisson brackets for the angular momentum [4, 5]

$$\{L_i, L_j\} = \sum_k \varepsilon_{ijk}L_k,$$

and the Hamiltonian

$$H = \frac{3}{16}L^2\sum_i \varepsilon_i\left[\left(\frac{L_i}{L}\right)^2-1\right]^2. \tag{5}$$

This circumstance is particularly interesting because, usually, the averaging procedure is not compatible with Hamiltonian structure. The system we have obtained is the integrable Hamiltonian one, but its exact solution is cumbersome. Therefore, we shall find a qualitative description of the system's motion and extensively use numerical simulation.

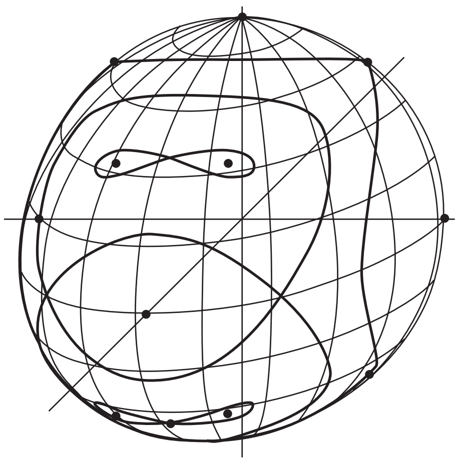

Fig. 2. Separatrix net corresponding to the graph of Type I on the sphere; the twin points correspond to the symmetry given by Eq. (6)

The important point is considering the stationary solutions to Eqs. (4) for which the right-hand sides turn out to be zero. They split into three parts S1, S2 and S3, determined by conditions on ε_i, as follows.

S1. No algebraic constraints imposed on ε_i:

a. $L_{10} = 0, \quad L_{20} = 0, \quad L_{30} \neq 0;$

b. $L_{10} = 0, \quad L_{20} \neq 0, \quad L_{30} = 0;$

c. $L_{10} \neq 0, \quad L_{20} = 0, \quad L_{30} = 0.$

S2. The constraints on $\mathbf{L}$ relaxed and linear constraints imposed on ε_i:

a. $L_{10} = 0, \quad L_{20} \neq 0, \quad L_{30} \neq 0, \quad \varepsilon_3 L_{20}^2 - \varepsilon_2 L_{30}^2 = 0;$

b. $L_{20} = 0, \quad L_{30} \neq 0, \quad L_{10} \neq 0, \quad \varepsilon_1 L_{30}^2 - \varepsilon_3 L_{10}^2 = 0;$

c. $L_{30} = 0, \quad L_{10} \neq 0, \quad L_{20} \neq 0, \quad \varepsilon_2 L_{10}^2 - \varepsilon_1 L_{20}^2 = 0.$

S3. Vector $\mathbf{L}$ subject to $L_{10} \neq 0, L_{20} \neq 0, L_{30} \neq 0$ and the quadratic constraints imposed on ε_i:

$$\frac{L_{10}^2}{\varepsilon_1 \varepsilon_2 - \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1} = \frac{L_{20}^2}{\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 - \varepsilon_3 \varepsilon_1} = \frac{L_{30}^2}{-\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1}.$$

It is worth noting that equations S2 involve the fulfilment of the inequalities $\varepsilon_2\varepsilon_3 > 0$, $\varepsilon_3\varepsilon_1 > 0$, and $\varepsilon_1\varepsilon_2 > 0$ for cases S2.a, S2.b, S2.c, respectively, whereas equations S3 involve

$$\begin{aligned} \varepsilon_1 \varepsilon_2 - \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1 &> 0, \\ \varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 - \varepsilon_3 \varepsilon_1 &> 0, \\ -\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1 &> 0. \end{aligned}$$

Linearizing Eqs. (4) at the stationary solutions and considering small fluctuations of $\mathbf{L}$ round them, we may study their stability. Some of them turn out to be centers and the others saddle points.

We may put this information in a graphic form by using the integral $L^2 = \mathrm{const}$, and consider the motion of $\mathbf{L}$ on a sphere of fixed radius, the integral of energy H taking appropriate values. Then the stationary solutions are fixed points as regards Eqs. (4), the stable and the unstable points are centers and saddle points, respectively, the separatrices being lines joining the fixed points. Together, they generate a graph on the sphere, having the fixed points as vertices and the separatrices as edges. It is important that the separatrices, i.e., the edges of the graph, are oriented according to the time t, so that the graph is the oriented one, and invariant with respect to the symmetry

$$\mathbf{R} \to -\mathbf{R}, \quad t \to -t. \tag{6}$$

Considering different values of ε_i, we obtain the following topological types of the graphs:

Type I. 7 centers and 6 saddles, ε_i being subject to the constraints:

$$\begin{aligned} \varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 &> 0, \\ \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 - \varepsilon_3\varepsilon_1 &> 0, \\ -\varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 &> 0. \end{aligned} \tag{7}$$

Type II. 5 centers and 4 saddle points; ε_i are not equal to zero, have the same sign, and at least one of Eqs. (7) is not true.

Type III. 3 centers and 2 saddle points, ε_i being subject to one of the following constraints: $\varepsilon_2\varepsilon_3 > 0$ and $\varepsilon_1\varepsilon_2 \leqslant 0$; $\varepsilon_3\varepsilon_1 > 0$ and $\varepsilon_2\varepsilon_3 \leqslant 0$; $\varepsilon_1\varepsilon_2 > 0$ and $\varepsilon_3\varepsilon_1 \leqslant 0$.

Type IV. 2 centers and 1 saddle point, ε_i being subject to one of the following constraints: $\varepsilon_1 = 0$ and $\varepsilon_2\varepsilon_3 \leqslant 0$; $\varepsilon_2 = 0$ and $\varepsilon_3\varepsilon_1 \leqslant 0$; $\varepsilon_3 = 0$ and $\varepsilon_1\varepsilon_2 \leqslant 0$.

Taking into account the homogeneous form of the constraints imposed on ε_i, we may visualize them on the projective plane corresponding to ε_i (see Fig. 3).

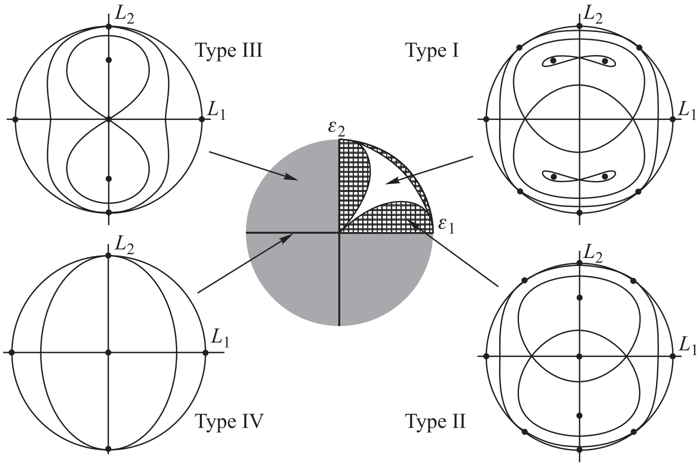

Fig. 3. Regions of ε_i corresponding to Types I–IV of the phase diagrams of the auxiliary system

It should be noted that we must check as to whether the solutions provided by Eqs. (4) agree with those given by original Eqs. (1) (see Fig. 4).

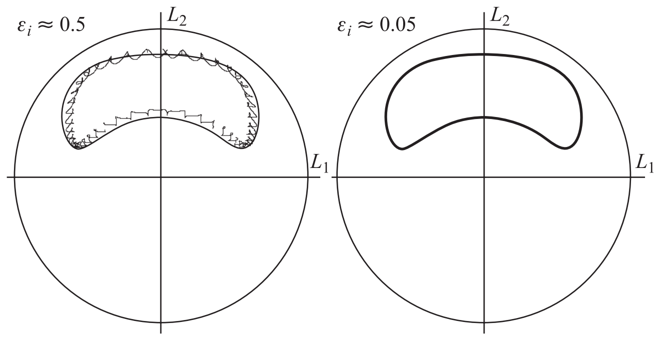

Fig. 4. Comparison of the solution to the initial equations for geodesics and the averaged equation given by the auxiliary system

CONCLUSION

The key point of the present investigation is the auxiliary Hamiltonian system, which can be considered as a reduction of the initial problem to a dynamical problem on specific configuration and phase spaces. Points of the new configuration space are geometrical objects, i.e., great circles, of the configuration space of the base, i.e., the standard sphere, so that we obtain a Hamiltonian system that describes the transformation of these objects. In analytical terms, one may consider it as an asymptotic reduction of the system of equations for orbits on a deformed sphere to that of the top, but with the Hamiltonian of the fourth order. The simplification we get in this way is substantial. Indeed, the Hamiltonian system for geodesics could be non-integrable, whereas the auxiliary system is totally integrable and described by a graph that comprises vertices, which correspond to stationary solutions, or almost closed orbits, and edges, which can be visualized as orbits joining them.

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