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КОМПЬЮТЕРНЫЕ ТЕХНОЛОГИИ В ФИЗИКЕ

FOUR-POINT TRANSFORMATION METHODS IN APPROXIMATION AND THE SMOOTHING PROBLEMS

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The main goal of the adaptive local strategy consists in reducing the complexity of computational problems. We propose a new approach to curve approximation and smoothing based on 4-point transformations or Discrete Projective Transforms (DPT). In the framework of DPT the variable point is related to three data points (accompanying points). The variable y ordinate is expressed via the convolution of accompanying y ordinates and weight functions that are defined as cross-ratio functions of four x coordinates. DPT has some attractive properties (natural norming, scale invariance, threefold symmetry, «4-point» orthogonality), which are useful in designing new algorithms. Diverse methods and algorithms based on DPT have been developed.

Основная цель адаптивной локальной стратегии состоит в понижении вычислительной сложности алгоритмов. В работе предлагается новый подход к аппроксимации и сглаживанию кривых, основанный на 4-точечных преобразованиях или дискретных проективных преобразованиях (ДПП). В рамках ДПП текущая точка связывается с тремя точками данных (сопровождающими точками). Текущая ордината y выражается через свертку сопровождающих y-ординат с весовыми функциями, определяемыми сложным отношением четырех x-координат. ДПП обладает рядом привлекательных свойств (естественная нормировка, масштабная инвариантность, тройная симметрия, «4-точечная» ортогональность), которые полезны при создании новых алгоритмов. На основе ДПП был разработан ряд разнообразных методов и алгоритмов.

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INTRODUCTION

Efficient computing methods for approximation and the smoothing problem are of great significance in the area of the development of information technologies. Fast processing of huge streams of data in a real time mode needs robust, flexible and rapidly computable algorithms. Investigations in this area are developed intensively. Such known methods as recursive least squares (RLS) of the third and higher degrees, splines, Kalman filtering (KF) have high computing complexity, especially in the case of smoothing of scattered data.

Smoothing algorithms oriented toward the real time mode require new, frequently conflicting, properties and restrictions: *stability to random errors, adaptability to input data, high speed of processing* for limited resources of storage and time. Tasks with such requirements arise everywhere, for example, in experimental nuclear physics, in pattern recognition, in technological processes managements, in digital signal processing, in mathematical techniques of calculations, etc.

A new approach to the problem of efficient computing of an optimal estimation $\hat{f}_*$ at a given point x_* using a polynomial model is proposed. We want to estimate an unknown signal from some noisy or not noisy data:

$$\{S\} = \{(x_k, \tilde{f}_k)\}_{k=1}^{N}, \quad x_k < x_{k+1}, \quad N \gg 4, \tag{1}$$

where $\tilde{f}_k = f(x_k) + e_k$, $e_k \sim$ iid $N(0, \sigma^2)$. If $\tilde{f}_*$ are obtained in numerical computations with small errors ($\sigma^2 \approx 0$), the problem of estimation of $\hat{f}_*$ is related to approximation (interpolation).

When the number (N) of samples is known, the least square estimation of coefficients $\mathbf{a} = [a_0, a_1, \ldots, a_n]^T$ in the standard polynomial model (n is the degree of the polynomial)

$$f(x) \approx P_n(x; \mathbf{a}) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + e \tag{2}$$

is written as

$$\hat{\mathbf{a}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \tilde{\mathbf{f}}, \tag{3}$$

where $\tilde{\mathbf{f}} = [\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_N]^T$ is a vector of measurements and $\mathbf{A}$ is the $N \times (n+1)$ regression matrix with elements $x_i^k, i = 1, 2, \ldots, N; k = 0, 1, \ldots, n$.

As is well known, the problem of computational complexity and stability of the task (1)–(3) becomes more complicated if n and σ^2 are increased. When errors are small, such problems are removed by using splines or the piecewise polynomial approximation. The smoothing methods (such as splines, nonparametric or stochastic approximation methods (SA) [1–4]) are used for noisy data. A number of algorithms and programs in this field can be found in modern program packages Matlab, Maple, SPlus, etc.

Different recurrent methods, such as autoregressive (AR), SA or RLS methods, are used when the data points (1) represent a time series. In these cases the vector $\hat{\mathbf{a}}$ and the inverse matrix $(\mathbf{A}^T \mathbf{A})^{-1}$ are determined by recursive procedures. It is known that RLS is restricted in practice when $n \geqslant 3$, because the computational complexity is proportional to n^2 [5].

1. $\alpha\beta$-PARAMETRIZATION

The goal is to design a DPT-polynomial model that is equivalent to $P_n(x)$ and possesses by *uniform approximation* on the interval, *small computational complexity*, *robustness*, *adaptability* and *controllability*. A polynomial with such properties can be constructed using 4-point transformations or DPT (Discrete Projective Transforms) that have been developed specially for these purposes [6, 8, 9]. It is necessary to note two essential moments in the structure of the polynomial model based on DPT.

First, DPT uses three reference points on the plane: $\{(x_i, r_i)\}$, $i = \overline{0, 2}$, $\prod_{i \neq j}(x_i - x_j) \neq 0$. These points are related to $f(x)$ or samples and are called *«a mark»* or *«companion points»*. They are common points for both $f(x)$ and $P_n(x)$.

Second, the reference points are used as parameters of the DPT-model: $r_i, i = \overline{0, 2}$ are coefficients of the quadratic part of the DPT-model and x_i are used for continuous parametrization of the power functions $\{x^n\}$, $n = 0, 1, 2, \ldots$

Insertion of the reference points into the polynomial model affects the uniformity of approximation and the computational complexity, whereas continuous parametrization of the model allows one to introduce various modes of control by the accuracy of approximation and the smoothing. These properties extend the limits of such classical methods as the least-squares method (LSM), RLS, splines, etc.

Both stability and computational complexity in Eqs. (2), (3) essentially depend on the choice of *basic functions*. In a number of cases the use of $\{x^k\}_{k=0}^n$ as basic functions results in an increase of the computational complexity. DPT-methods enhance stability to errors and the computation speed due to: a) *parametrization of the basic functions*; b) *reducing the model degree* and c) *recursive computation of parameter estimations*.

1.1. What are Four-Point Transformations? In the 4-point transformation methods [6,9] current coordinates $(x_\tau, y_\tau) \in R^2$ are considered always *in the aggregate* with coordinates of the reference points. In such a quadruplet the parameters α and β are defined as $\tau = x_\tau - x_0$ at $\tau = \alpha, \beta$, where $x \equiv x_\tau$ (Fig. 1). The relation between the point (x, y) and the rest points in the tetrad is established by convolution of two vectors $\mathbf{v} = [v_1, v_2, v_3]^T$ and $\mathbf{y} = [y_\alpha, y_\beta, y]^T$:

$$y^\triangleleft = \mathbf{v}^T\mathbf{y} = y_\alpha v_1 + y_\beta v_2 + y v_3, \tag{4}$$

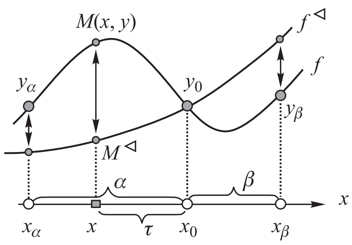

Fig. 1. The result of $M \leftrightarrow M^\triangleleft$ transformation

where the symbol $\triangleleft$ denotes the direct DPT. Coordinates $\{v_i\}$ of the weight vector $\mathbf{v}$ depend on τ and parameters α, β. Four points in the quadruplet are related in accordance with the special cross-ratio algorithm: $[13]/[24] : [23]/[14]$, where $[ij] = x_j - x_i$ [6]. Depending on the indexing of points in the tetrad $\{x_0, x_\alpha, x_\beta, x\}$ this algorithm generates sets of various weight functions. If the first point is fixed, we obtain six functions, three of which are different. We shall denote these functions as $v_i(\tau; \alpha, \beta)$, and $w_i(\tau; \alpha, \beta)$, $i = 1, 2, 3$. v_i are obtained from the tetrad $\{0\alpha\beta\tau\}$, whereas w_i are got using the tetrad $\{\tau\alpha\beta 0\}$:

$$v_1 = \frac{\tau\beta}{(\tau-\alpha)\gamma}, \quad v_2 = \frac{-\tau\alpha}{(\tau-\beta)\gamma}, \quad v_3 = \frac{\alpha\beta}{(\tau-\alpha)(\tau-\beta)}, \quad \sum_{i=1}^{3} v_i = 1, \tag{5}$$

$$w_1 = \frac{-\tau(\tau-\beta)}{\alpha\gamma}, \quad w_2 = \frac{\tau(\tau-\alpha)}{\beta\gamma}, \quad w_3 = \frac{(\tau-\alpha)(\tau-\beta)}{\alpha\beta}, \quad \sum_{i=1}^{3} w_i = 1, \tag{6}$$

where $\gamma = \beta - \alpha$.

In Eqs. (4), (5) for $\tau = 0$ we have $y_0^\triangleleft = y_0$, whereas values $y_\alpha^\triangleleft$, $y_\beta^\triangleleft$ are defined by passage to the limits in Eq. (5) at $\tau \to \alpha$ and $\tau \to \beta$:

$$y_\alpha^\triangleleft = -\frac{\alpha\beta}{\gamma} y_\alpha^{\prime} + \frac{\alpha^2}{\gamma^2}(y_\beta - y_\alpha) + y_\alpha \quad \text{and} \quad y_\beta^\triangleleft = \frac{\alpha\beta}{\gamma} y_\beta^{\prime} - \frac{\beta^2}{\gamma^2}(y_\beta - y_\alpha) + y_\beta.$$

From Eq. (4) the inverse transformation IDPT is defined as

$$y = \mathbf{w}^T\mathbf{y}^\triangleleft = y_\alpha w_1 + y_\beta w_2 + y^\triangleleft w_3, \tag{7}$$

where $\mathbf{y}^\triangleleft = [y_\alpha, y_\beta, y^\triangleleft]^T$ and $\mathbf{w} = [w_1, w_2, w_3]^T$.

Functions w_i are *quadratic* functions with respect to τ and *fractional rational* functions with respect to continuous parameters α and β.

Equations (4), (7) define direct and inverse 4-point transformations, based on the reference points and weight functions $\mathbf{v}$ and $\mathbf{w}$. From a geometrical point of view, DPT and IDPT transform the point $M(x, y)$ on the curve f (original) into the point $M^{\triangleleft}(x, y^{\triangleleft})$ of another curve $f^{\triangleleft}$ (image) and vice versa, leaving the point $M_0(x_0, y_0)$ immovable (Fig. 1). The shape of the curve $f^{\triangleleft}$ is changed with respect to the shape of f, since the denominators of functions $\{v_i\}$ depend quadratically on parameters and variables. For example, points on straight lines and quadratic parabolas are transformed into the horizontal $y^{\triangleleft} = y_0 = \text{const}$.

1.2. Properties of DPT. Let us note some properties of DPT.

1°. $c^{\triangleleft} = c$, where c is a constant, since $\sum_{i=1}^{3} cv_i = c\sum_{i=1}^{3} v_i = c \cdot 1 = c$.

2°. *Decrease the degree of monomials* $\{x^n\}$ by two. From (4) and (5) using $\{0\alpha\beta x\}$ and $\mathbf{y} = [\alpha^n, \beta^n, x^n]^T$, $\alpha\beta\gamma \neq 0$, $n = 0, 1, 2, \ldots$ we obtain the 4-point transformation for functions $\{x^n\}$ [6]:

$$[x^n]^{\triangleleft} = \alpha\beta x \underbrace{\left[\sum_{i=1}^{n-1} \alpha^{i-1} \sum_{k=1}^{n-i-1} \beta^{k-1} x^{n-i-k-1}\right]}_{z_{n-3}(x;\alpha,\beta)} = \alpha\beta x z_{n-3}(x; \alpha, \beta), \tag{8}$$

whence it follows that

$$1^{\triangleleft} = 1, \quad [x]^{\triangleleft} = 0, \quad [x^2]^{\triangleleft} = 0, \quad [x^3]^{\triangleleft} = \alpha\beta x, \ldots$$

The degree of polynomial $z_{n-3}(x; \alpha, \beta)$ with respect to x is less than the degree of x^n by three.

3°. *Orthogonal property* of the vector $\mathbf{v}$ to vectors forming from ordinates of points situated on a straight line or a quadratic parabola. From **1°** and **2°** using the linear combination $y = ax^2 + bx + c$ one can obtain: $y^{\triangleleft} = \mathbf{v}^T\mathbf{y} = y_0$, where $y_0 \equiv c$. From this it follows that $\mathbf{v}^T\Delta\mathbf{y} = 0$, where $\Delta\mathbf{y} = [y_\alpha - y_0, y_\beta - y_0, y - y_0]^T$.

The orthogonal property is useful for transformation data points scattered around a straight line or a quadratic parabola, since transformation of the difference $\Delta\tilde{y} = \tilde{y} - \tilde{y}_0$ leads to the error equation: $\Delta\tilde{y}^{\triangleleft} = \mathbf{v}^T(\Delta\mathbf{y} + \Delta\mathbf{e}) = \mathbf{v}^T\Delta\mathbf{y} + \mathbf{v}^T\Delta\mathbf{e} = 0 + \mathbf{v}^T\Delta\mathbf{e} = \varepsilon$, where $\Delta\mathbf{e} = [e_\alpha - e_0, e_\beta - e_0, e - e_0]^T$. Thus, $\Delta\mathbf{e}$ is transformed into ε via square-law denominators of $\mathbf{v}$. In this case, the linear or quadratic systematization in $\Delta\mathbf{e}$ is removed.

1.3. Parametrization of the Power Functions $\{x^n\}$. To achieve *$\alpha\beta$-parametrization* of the power functions $\{x^n\}$, we use the inverse transformation (IDPT). To do this, we substitute x^n, α^n, β^n and $[x^n]^{\triangleleft}$ from Eq. (8) into Eq. (7) in place of y, y_α, y_β and $y^{\triangleleft}$, respectively. We get

$$\begin{aligned} x^n &= \alpha^n w_1 + \beta^n w_2 + [x^n]^{\triangleleft} w_3 = \alpha^n w_1 + \beta^n w_2 + \alpha\beta x z_{n-3}(x; \alpha, \beta) w_3 = \\ &= \alpha^n w_1 + \beta^n w_2 + \underbrace{x(x-\alpha)(x-\beta)}_{Q} z_{n-3}(x; \alpha, \beta) = \alpha^n w_1 + \beta^n w_2 + \underbrace{Q z_{n-3}(x; \alpha, \beta)}_{s_n} = \\ &= \alpha^n w_1 + \beta^n w_2 + s_n(x; \alpha, \beta), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{9}$$

Thus, in Eq. (9) the elementary power functions x^n are parameterized by α, β continuously and are partitioned into two parts: the square-law parabola $(\alpha^n w_1 + \beta^n w_2)$ and the polynomial

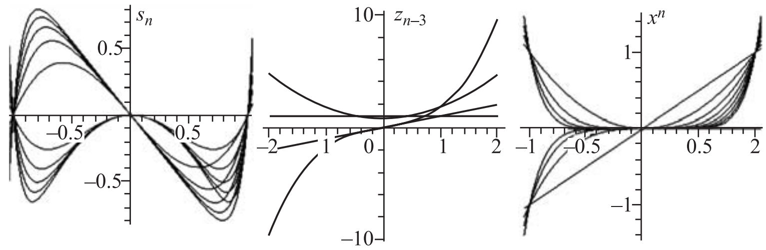

Fig. 2. Plots of s_n, z_{n-3} and x^n

$s_n = Qz_{n-3}(x;\alpha,\beta)$, where $Q(x;\alpha,\beta) = x(x-\alpha)(x-\beta)$ is a «zeroed» cubic parabola. Figure 2 shows plots of s_n, z_{n-3} and x^n.

Remark 1. Parameters α and β in Eq. (9) are chosen arbitrarily provided $\alpha\beta\gamma \neq 0$.

1.4. Parametrization of the Polynomial $P(x;\mathbf{a})$**.** Polynomials $z_{n-3}(x;\alpha,\beta)$ in Eq. (9) are determined by recurrence [9]:

$$z_j = (x+\alpha)z_{j-1} - \alpha x z_{j-2} + \beta^j, \quad z_{-1} = z_{-2} = 0; \quad j = 0,1,\ldots,n-3 \tag{10}$$

and represent *elementary symmetric polynomials*. For $n = 3,4,5$ we obtain

$$z_0 = 1, \quad z_1 = x+\alpha+\beta, \quad z_2 = x^2+\alpha^2+\beta^2+\alpha x+\beta x+\alpha\beta,\ldots$$

Substituting the right-hand side of Eq. (9) into Eq. (2) with account of Eq. (10) and the property $\mathbf{1}^\circ$ of DPT, we have

$$\begin{aligned} P(x) &= \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} a_i[\alpha^i w_1 + \beta^i w_2 + s_i] = Aw_1 + Bw_2 + \sum_{i=0}^{n} a_i s_i = \\ &= \underbrace{Aw_1 + Bw_2 + a_0}_{\Pi(x;\alpha,\beta,\mathbf{r})} + Q\sum_{i=0}^{n-3} c_i z_i(x;\alpha,\beta) = \Pi(x;\alpha,\beta,\mathbf{r}) + QU_{n-3}(x;\alpha,\beta), \end{aligned} \tag{11}$$

where $A = \sum_{i=0}^{n} a_i\alpha^i, B = \sum_{i=0}^{n} a_i\beta^i$, $\mathbf{r} = [P_\alpha, P_\beta, P_0]^T$, $P_* \equiv P(*)$ and c_i are coefficients. Considering that $\Pi(x;\alpha,\beta,\mathbf{r}) = \mathbf{w}^T\mathbf{r} = P_\alpha w_1 + P_\beta w_2 + P_0 w_3$, the values of the polynomial $U_{n-3}(x;\alpha,\beta) = [P(x) - \Pi(x;\alpha,\beta,\mathbf{r}))]/Q$ at points 0, α and β can be found by suitable limits: $U_{n-3}(\rho) = \lim\limits_{x\to\rho} U_{n-3}(x;\alpha,\beta), \rho = \alpha,\beta,0$, i.e.,

$$\begin{aligned} U_{n-3}(\alpha) &= (\alpha\gamma)^{-1}[(\alpha^2 P_\beta - \gamma^2 P_0)/(\alpha\beta\gamma) - P'_\alpha], \\ U_{n-3}(\beta) &= (\beta\gamma)^{-1}[(\beta^2 P_\alpha - \gamma^2 P_0)/(\alpha\beta\gamma) + P'_\beta], \\ U_{n-3}(0) &= (\alpha\beta)^{-1}[(\alpha^2 P_\beta - \beta^2 P_\alpha)/(\alpha\beta\gamma) + P'_0]. \end{aligned} \tag{12}$$

Hence, Eqs. (11), (12) present *a continuous $\alpha\beta$-parametrization* of the standard polynomial $P(x;\mathbf{a})$ in the form of the sum of a square-law parabola $\Pi(x;\alpha,\beta,\mathbf{r})$, passing through the reference points and the reduced polynomial $U_{n-3} = \sum_{i=0}^{n-3} c_i z_i(x;\alpha,\beta)$ multiplied by the «zeroed» cubic parabola Q.

So, we get a new decomposition of $f(x) \in C_{[\alpha,\beta]}$ using w_i and basis $\{s_k\}$:

$$f(x) \approx \mathbf{w}^T\mathbf{r} + \sum_{k=3}^{n} c_k s_k(x;\alpha,\beta), \tag{13}$$

where $s_3 = Q$ and $\mathbf{r} = [r_\alpha, r_\beta, r_0]^T$ are ordinates of the reference points. Polynomials s_k have a structure of kth order *monosplines*, which play, in a certain sense, the same role in approximation theory as Chebyshev polynomials do in the classical function approximation theory. From Fig. 2 one can see that polynomials $\{s_k\}$ have *a better approximation quality* than monomials $\{x^k\}$: when x tends to zero they also tend to zero by a linear or quadratic law, depending on the parity of k.

1.5. Expansion of $P_n(x)$ by $Q^i(x;\alpha,\beta)$. It should be pointed out that if we apply Eq. (11) sequentially for «reduced» polynomials $U_{n-3i}(x;\alpha,\beta)$

$$P_n(x) = \Pi_0(x;\alpha,\beta,\mathbf{r}_0) + Q[\Pi_1(x;\alpha,\beta,\mathbf{r}_1) + \ldots + Q[\Pi_i(x;\alpha,\beta,\mathbf{r}_i)]\ldots]],$$

we achieve expansion of the standard polynomial $P_n(x)$ by degrees of Q^i with multipliers $\Pi_i, i = 0, 1, 2, \ldots, k \ll n$:

$$P_n(x) = \sum_{i=0}^{k} Q^i \Pi_i(x;\alpha,\beta,\mathbf{r}_i), \quad k \ll n, \tag{14}$$

where $Q = \tau(\tau-\alpha)(\tau-\beta)$, $\Pi_i(x;\alpha,\beta,\mathbf{r}_i) = r_{\alpha i}w_1 + r_{\beta i}w_2 + r_{0i}w_3$, and $\mathbf{r}_i$ are unknown reference points on polynomials U_{n-3i}.

2. THE STRUCTURE AND PROPERTIES OF THE REDUCED MODEL

The properties of the standard polynomial model (2) are changed essentially upon $\alpha\beta$-parametrization of $P_n(x)$.

First, the choice of the reference coordinates from data points (1) provides a natural attaching to the curve f. The ordinates r_* are chosen depending on σ^2 as $r_* \equiv f_*$ or $r_* = \hat{f}_*$, where $\hat{f}_*$ is the local estimation of neighboring ordinates.

Second, the parameterized model (13) consists of two parts: the fixed part ($\mathbf{w}^T\mathbf{r}$) and the free part ($\mathbf{s}^T\mathbf{c}$), i.e.,

$$f(x;\alpha,\beta;\mathbf{r}) = \mathbf{w}^T\mathbf{r} + Q\mathbf{z}^T\mathbf{c} + e, \tag{15}$$

where $\mathbf{z}^T = [z_0, z_1, \ldots, z_m]$ is a «reduced» basis and $\mathbf{c} = [c_0, c_1, \ldots, c_m]^T$ is the vector of unknown coefficients ($m = (n-3) < N$).

This makes possible calculation using continuous parameters α, β as variables and significantly extends the frames of algorithmization of calculations from the viewpoint of construction of recursion schemes. It is necessary to note that Eq. (15) extends the Clenshaw–Hayes polynomial presentation [7].

Third, reduction of the power of basis functions permits one to decrease the computational complexity of task and to increase the robustness of algorithms.

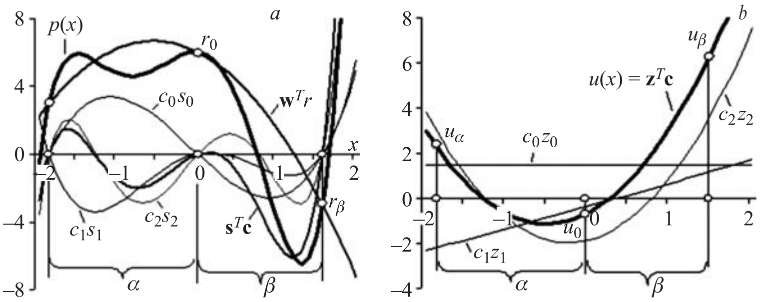

Fig. 3. Decomposition of the 5th order polynomial $p(x)$ via $w_j(x;\alpha,\beta)$ and $s_j(x;\alpha,\beta), j=1,2,3$ (*a*). Reducing $p(x)$ to the square-law parabola u (*b*)

Dividing Eq. (15) by $Q \neq 0$, one can transform the data points $\{\tilde{f}(x_k)\}$ into $\{\tilde{u}(x_k)\}$:

$$\tilde{u}(x) = (\tilde{f}(x) - \mathbf{w}^T\mathbf{r})/Q(x;\alpha,\beta) = u(x) + \varepsilon(x). \tag{16}$$

To approximate the transformed data $\tilde{u}(x)$ a polynomial of lower degree is required. In this case the dimension of the initial task is diminished:

$$\tilde{u}(x) = \mathbf{z}^T\mathbf{c} + \varepsilon = \sum_{i=0}^{n-3} c_i z_i + \varepsilon, \tag{17}$$

where $\varepsilon(x) = e(x)/Q(x;\alpha,\beta)$. Figure 3 shows an example of reducing the 5th order polynomial $p_5(x;\alpha,\beta,\mathbf{r})$ to the quadratic polynomial

$$u = c_0 \overbrace{(1)}^{z_0} + c_1 \overbrace{(x+\alpha+\beta)}^{z_1} + c_2 \overbrace{(x^2 + (\alpha+\beta)x + \alpha\beta + \alpha^2 + \beta^2)}^{z_2}.$$

2.1. Stability of $\tilde{f} \to \tilde{u}$ Transformation to Errors. In accordance with Eq. (16), the errors $e(x)$, $e(\alpha)$, $e(\beta)$, $e(0)$ are transformed into $\varepsilon(x)$, $\varepsilon(\alpha)$, $\varepsilon(\beta)$, $\varepsilon(0)$ via the denominator of functions $1/Q(x;\alpha,\beta)$ and $w_i/Q(x;\alpha,\beta)$.

Figure 4 shows the behaviour of ε_{*n} in moving coordinates (the origin 0_n is shifted to the right by the step $h > 0$ with respect to the unmoved curve). If $\alpha_n < \beta_n < 0_n < x_n$, then $Q_n(x_n;\alpha_n,\beta_n) = h^3(n+2)(n+1)$, where $\alpha_n = -(n+1)h, \beta_n = -nh$ and $x_n = h$;

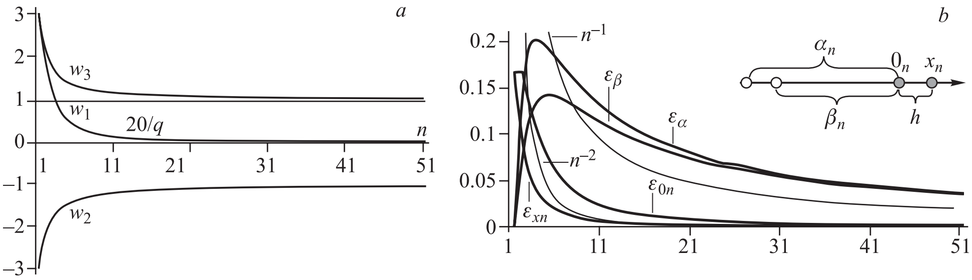

Fig. 4. Weight functions (*a*) and the behaviour of ε_{*n} in moving coordinates (*b*)

$n = 1, 2, \ldots$ (Fig. 4, at upper right). Graphs of ε_{xn} and ε_{0n} vary as n^{-2}. Both errors ε_α and ε_β (reference points r_α and r_β) approach n^{-1}. Functions n^{-1} and n^{-2} are shown for comparison. Thus, Eqs. (9)–(14) and (16) allow one *to reduce the dimension* of the task (1)–(3) as well as *to provide stability* of computations with respect to input errors. The continuous parameters α and β are control parameters of the computational process.

3. LOCAL CUBIC APPROXIMATION AND SMOOTHING

The above approach has been used for the development of new algorithms for piecewise-cubic approximation and smoothing [10, 11, 20]. Taking $n = 3$ in Eq. (13), we obtain the three-point spline model (TPS):

$$S = \mathbf{w}^T \mathbf{r} + \theta Q(x; \alpha, \beta). \tag{18}$$

This model depends upon three fixed ($\mathbf{r}$) and one free (θ) parameters. Fixed parameters $\mathbf{r}$ are used for relating the model to input data, θ is an unknown parameter. Based on the TPS-model, the first order RLS-procedure has been obtained for estimation of the θ-parameter using the following recursion:

$$\hat{\theta}_n = \hat{\theta}_{n-1} + K_n[\tilde{f}_n - \Pi_n(h; \mathbf{w}_{in}, \hat{\mathbf{r}}_n) - \hat{\theta}_{n-1} Q_n(h; \alpha_n, \beta_n)], \quad \theta_0 = 0, \tag{19}$$

where $n = 1, 2, \ldots$; $K_n = Q_n / \sum_{k=1}^{n} Q_k^2$ is the amplification factor.

Equation (19) has been used for developing a real time-oriented algorithm LOCUS [10] and an algorithm for automatic knot detection in piecewise-cubic approximation and smoothing [11]. The free knots optimization problem is a very difficult nonlinear problem [19] and it is important for applications. The main goal is to find an optimal subdivision so that the errors over the subintervals are as small as possible. This problem is closely related to the

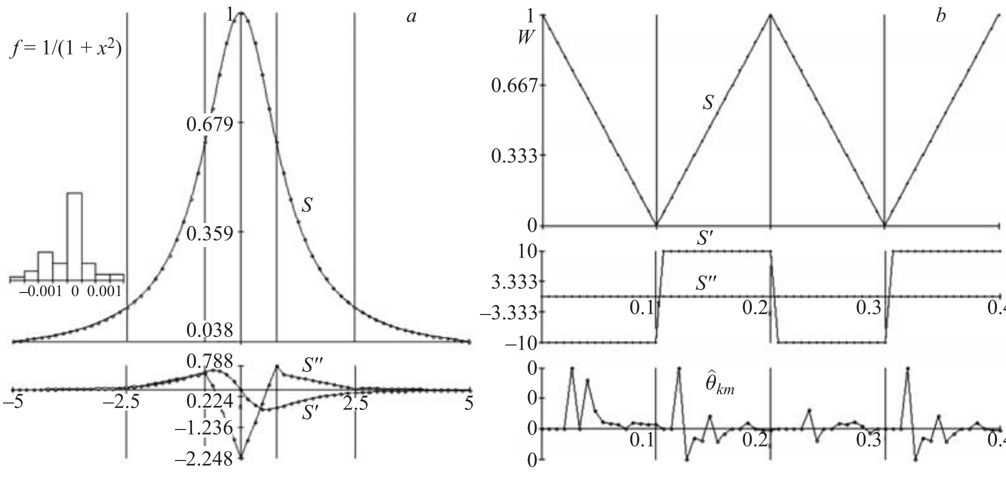

Fig. 5. APCA-approximation data without noise: *a*) the Runge function; *b*) W function

optimum choice of knots in approximation by cubic splines. For the automatic tracking of a cubic segment of a curve the criterion of uniformity of the third derivative of the cubic model (18) is used.

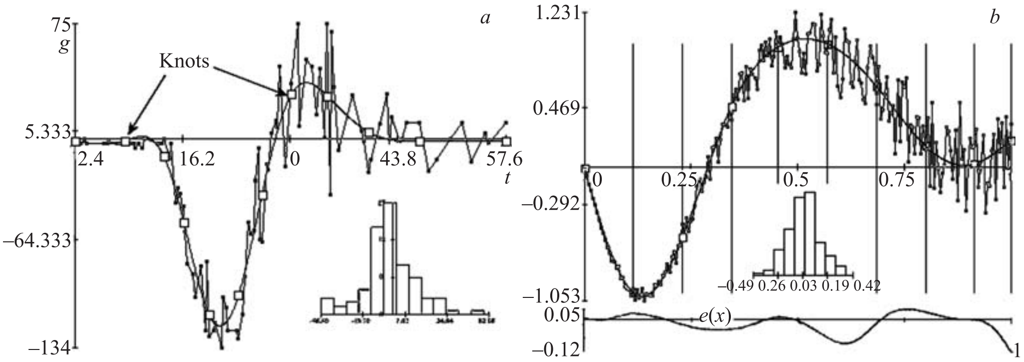

Fig. 6. APCA-smoothing of noisy data: *a*) «motorcycle» data [2]; *b*) the test function [3]

A Windows application APCA (Autotracking Piecewise-Cubic Approximation) was developed [14,15]. The efficiency of the method and the algorithm is confirmed by processing real scattered data and approximation of complex curves presented by data points with $\sigma^2 \approx 0$. An example of results for the Runge and W-shape functions segmentation is shown in Fig. 5 [11]. Figure 6 illustrates new APCA-smoothing results [20, 21] of the so-called motorcycle data from the book [2] and simulated data $\tilde{y}_i = \sin(2\pi(1 - x_i)^2 + x_i\epsilon_i)$, $x \in [0, 1]$; $i = \overline{1, n}$, $n = 200$; $\epsilon_i \sim N(0, 1)$ from [3]. The distributions of residuals (histograms) and the errors $e(x_i) = s(x_i) - \sin(2\pi(1 - x_i^2))$ are also shown. Vertical lines show the knot-positions detected by APCA.

CONCLUSIONS

We have proposed a 4-point methodology for reducing the polynomial degree and for parametrization of $P(x; \mathbf{a})$ using reference coordinates as continuous parameters for control by the accuracy of approximation or smoothing. This approach extends the framework of known classical methods and improves the quality and efficiency of approximation algorithms. A number of methods and algorithms have been developed using DPT: the adaptive projective filters for track finding [8]; DPT-function parametrization for *uniform* approximation over the whole interval [9]; polynomial approximation $f(x) \in C^{(n)}_{[a,b]}$, $f(x) \in L_2$ [9]. Comparison of DPT approximation with Chebyshev and Pade approximation has been done. While the Pade-approximants behave better near zero, the DPT-approximants give smaller maximum errors over the whole interval, however, they seem to be greater than the maximum error of the Chebyshev-approximants. The quality of the DPT-approximants greatly depends on the choice of derivative points [12]; TPS-model and two-stage recursive algorithm LOCUS-P [10] for piecewise-cubic smoothing.

The estimate $\hat{\theta}$ is determined by the recursive least squares procedure, with the amplification factor $K_n(\alpha, \beta) \sim O(n^{-3})$, where α, β are smoothing parameters; the 9-point model for surface smoothing [17]; in [13,18] the DPT of polynomials and the assessment of the polynomial degree were studied; the piecewise-cubic algorithm with auto detection of knots [11,21] (Figs. 5, 6); the papers [20] and [21] represent successful smoothing of strongly noising data by cubic splines with free knots based on the model (18), etc.

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