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CASIMIR ENERGY CALCULATIONS FOR CHERN–SIMONS SURFACES AND DIELECTRIC PLATES WITHIN THE FORMALISM OF LATTICE QUANTUM FIELD THEORY

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A new method based on the Monte Carlo calculation on the lattice is proposed to study the Casimir effect in the noncompact lattice QED. This method can be used for Chern–Simons surfaces (thin metal films) and for dielectric plates.

Предложен новый метод для вычисления энергии Казимира, базирующийся на монте-карловских вычислениях в некомпактной решеточной электродинамике. Этот метод может применяться для черн-саймонсовских поверхностей (тонких металлических пленок) и для диэлектрических пластин.

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1. INTRODUCTION AND GENERAL MOTIVATION

During the last few years the Casimir effect has attracted much attention due to the great experimental and theoretical progress in studying of this phenomenon. This macroscopic quantum effect plays a crucial role in nanophysics, micromechanics, quantum optics, condensed matter physics, material science, and it is also very important for different models of boundary states in hadron physics, heavy-ion collisions, and cosmology.

Nowadays, a lot of theoretical methods for calculation of the Casimir effect were proposed. Various approximate methods (like the proximity force approximation method [1, 5]) are used in the case of curved surfaces. There has been a large progress in this approximate method in the last few years due to the possibility to take into account the first terms of the asymptotic expansion for small separation between bodies [2]. Also, several exact methods were developed recently for the Casimir effect calculation. One of them is based on the Green function method [3]. In this method the full Casimir force on a body is expressed in terms of the mean electromagnetic stress tensor $\langle T_{ij} \rangle$ and its components can be obtained from Green's functions. Evaluation of the Green function is the standard task in various electromagnetic problems, so many standard techniques become applicable for the Casimir calculations in this case. Another interesting method is multipole expansion of the Casimir interaction [4]. The

Casimir energy is obtained in this method as an interaction between multipoles, generated by quantum current fluctuations.

In our work we propose a new numerical method for the Casimir energy calculation based on the Monte Carlo technique in the lattice quantum field theory. This approach can be useful because many effective numerical algorithms were developed for the Monte Carlo calculations in lattice QFT. These methods are rather simple and very suitable for parallel calculations, so it will be interesting to apply them for a new problem.

A crucial obstacle on the way of the realization of any Casimir problem on the lattice is the following. If the Casimir energy is a response of the vacuum to the presence of the boundary, what is «the boundary» in terms of lattice formalism? In other words, what is an observable quantity corresponding to such a boundary? In fact, the answer is nontrivial and leads to the definition of two new observational quantities in lattice field theory.

In Sec. 2 we will discuss the Chern–Simons boundary condition (it can be used as a model for thin metal films). In Sec. 3 we will consider Wilson «bag» and lattice description of Casimir effect in Maxwell–Chern–Simons theory. Section 4 will be devoted to the description of dielectric and in Sec. 5 we will discuss the continuous limit of lattice calculations.

2. CHERN–SIMONS BOUNDARY CONDITIONS AND CASIMIR EFFECT

Casimir effect is a response of the vacuum to boundary condition. The spectrum of vacuum fluctuations depends on the boundary conditions. Changing of the boundary conditions leads to changing of the spectrum of vacuum fluctuations and so to generating of the corresponding Casimir force on the boundary. In the standard quantum field theory formalism, such changing of spectrum of vacuum fluctuations can be described, for example, by means of the Green function method [1]. This approach is a very powerful tool for studying many essential Casimir tasks [1]. Unfortunately, the application of this analytical method to the case of more complicated shape of the boundary surfaces is not so easy due to calculation difficulties. Our aim is the creation of the numerical method for the Casimir effect calculation directly from the quantum field theory action. The lattice formalism looks very attractive for this role but manifestly we cannot base in our approach on the separation of vacuum modes corresponding to boundary from the full spectrum of vacuum fluctuation. In lattice formalism we work in Euclidean space and deal with full spectrum of vacuum fluctuations and cannot easily snatch out vacuum fluctuations corresponding to some boundary conditions. We need some very delicate approach for separation of vacuum fluctuation modes that preserve gauge invariance of our lattice formalism. Fortunately, such an approach to Casimir problem was proposed recently [8].

This approach is based on a very elegant idea coming from some unique properties of the Chern–Simons action in three dimensions [8]. Let us consider electromagnetic fields in $3+1$ dimensions with the Maxwell action and additional Chern–Simons action given on three-dimensional integral on the boundary surface S:

$$S = -\frac{1}{4}\int d^4x\ F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}\oint d^3s\varepsilon^{\sigma\mu\nu\rho}n_\sigma A_\mu(x)F_{\nu\rho}(x), \tag{1}$$

where $\varepsilon^{\sigma\mu\nu\rho}$ is the Levi–Civita tensor and n_σ is the normal vector to the boundary surface S; λ is a real parameter.

Let us consider now the simplest form of boundary surface S, namely two parallel infinite planes placed from each other at the distance R. The Chern–Simons formulation of this canonical Casimir problem was studied analytically in a series of works [9]. We will use this analytical answer for the fitting of our numerical data.

In the case of plane form of the boundary surface S, the Chern–Simons action in (1) has the following form:

$$S_{\mathrm{CS}} = \frac{\lambda}{2}\int (\delta(x_3) - \delta(x_3 - R))\varepsilon^{3\mu\nu\rho} A_\mu * (x) F_{\nu\rho}(x)\, d^4x,$$

where, in our formulation of this Casimir problem, normal vectors to the planes are turned in the opposite directions. This choice of the normal vector orientation corresponds to our renormalization procedure based on the connection between open and closed Casimir problems.

If the parameter λ is small, electromagnetic fields obviously do not feel any boundary and are free. What happens if the parameter λ becomes large and tends to infinity and fields dynamics on the boundary surface S is determined by Chern–Simons action? Let us consider the equation of motion obtained from the action (1):

$$\partial_\nu \partial_\nu A^\mu + \lambda(\delta(x_3) - \delta(x_3 - R))\varepsilon^{3\sigma\nu\rho} A_\sigma \partial_\nu A_\rho = 0. \qquad (2)$$

At $\lambda \to \infty$, it is easy to obtain from (2) corresponding boundary conditions on the surface S:

$$E_\parallel|_S = 0, \quad H_n|_S = 0,$$

where H_n and $E_\parallel$ are normal and longitudinal components of magnetic and electric fields, correspondingly. These conditions mean the nulling of the energy flux of the electromagnetic field through the surface.

The analytical answer of Casimir energy per unit area for two planes given by the Green function method is the following [9]:

$$E_{\mathrm{Cas}} = -\frac{\pi^2}{720R^3} f(\lambda),$$

where function $f(\lambda)\colon \lim_{\lambda\to\infty} f(\lambda) = 1$ can be written as

$$f(\lambda) = \frac{90}{\pi^4}\mathrm{Li}_4\left(\frac{\lambda^2}{\lambda^2+1}\right).$$

Here the polylogarithm function $\mathrm{Li}_4(x)$ is defined as

$$\mathrm{Li}_4(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^4} = -\frac{1}{2}\int_0^\infty k^2 \ln(1 - x\,\mathrm{e}^{-k})\, dk.$$

3. WILSON «BAG» AND LATTICE DESCRIPTION OF CASIMIR EFFECT IN MAXWELL–CHERN–SIMONS THEORY

In our work we use the four-dimensional hypercubical lattice and the action for the noncompact $U(1)$ QED:

$$S = \frac{\beta}{2} \sum_x \sum_{\mu<\nu} \theta^2_{p,\mu\nu}(x),$$

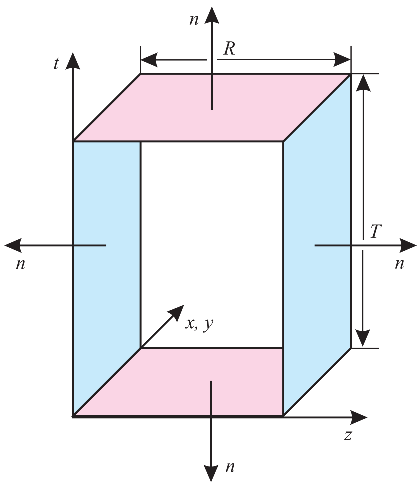
where the link and plaquette variables are defined as

$$\theta_{l,\mu}(x) = e\,a\,A_\mu, \quad \theta_{p,\mu\nu}(x) = \triangle_\mu \theta_{l,\nu}(x) - \triangle_\nu \theta_{l,\mu}(x),$$
$$\triangle_\mu \theta_{l,\nu}(x) = \theta_{l,\nu}(x + \hat{\mu}) - \theta_{l,\nu}(x).$$

Here a is a lattice step and the parameter $\beta = 1/e^2$. Physical quantities are calculated in the lattice formalism by means of field configuration averaging, where the field configurations (the set of all link variables) are generated with the statistical weight e^{-S}.

We have clarified in the previous section that the additional Chern–Simons action describes the Casimir effect. In order to find a lattice description of the Casimir interaction between boundary surfaces, let us consider Wilson loop, which describes the interaction of charged particles. Wilson loop can be written in QED in Euclidean time as

$$W_C = \exp\left(ig \oint_C A_\mu dx_\mu \right) = \exp\left(i \int J_\mu A_\mu dx^4 \right). \tag{3}$$



Wilson bag for two plane surfaces

The exponent in (3) is the additional term to the action. This term describes the interaction of the field A_μ with the current $J_\mu(x) = g \oint_C \delta(x - \xi)\, d\xi_\mu$ of charged particle. Configuration averaging of Wilson loop $\langle W(R,T) \rangle$ (where R and T are dimensions of the loop) converges in Euclidean time in the limit $T \to \infty$ to

$$\langle W(R,T) \rangle \to C\ \mathrm{e}^{-V(R)T},$$

where $V(R)$ is the energy of interaction between charged particles. The same method can be used for calculation of Casimir energy by means of Chern–Simons action.

Analogously to the description of charged particles interaction by 1D integral along Wilson loop, we will describe the Casimir interaction of surfaces by corresponding 3D integral. The first problem is that for stationary objects the action (2) is an integral from $t = -\infty$ to $t = \infty$, so by analogy to Wilson loop, we should enclose the surface of the integration in t direction. The integration surface for two planes is shown in the figure. This closing procedure can be performed both for plane surfaces and for any curved surface in three-dimensional space. As the result of this procedure, so-called Wilson bag [6, 7] can be obtained. It can be written as

$$\exp\left(i\lambda \oint_{\Sigma} \varepsilon_{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma} dS_\mu\right),$$

where Σ is closed three-dimensional surface in four-dimensional space-time. The final conclusion is that Wilson bag is (by analogy to Wilson loop) an observable quantity which gives us Casimir energy of the objects, defined by the surface of integration. For two planes we will calculate the following object:

$$W_{\text{bag}}(R,T) = \mathrm{e}^{\,i\lambda S(R,T)},$$

where

$$S(R,T) = \int_0^T dt \iiint dx\, dy\, dz\, (\delta(z-R) - \delta(z))\varepsilon_{3\nu\rho\sigma} A_\nu F_{\rho\sigma} + \\ + \int_0^R dz \iiint dx\, dy\, dt\, (\delta(t-T) - \delta(t))\varepsilon_{4\nu\rho\sigma} A_\nu F_{\rho\sigma}.$$

And in Euclidean time in the limit $T \to \infty$

$$\langle W_{\text{bag}}(R,T)\rangle \to C\,\mathrm{e}^{-E_{\text{Cas}}(R)T}.$$

The second problem is to rewrite the Wilson bag in terms of lattice objects (links and plaquettes). The product $A_\nu F_{\rho\sigma}$ can be exactly constructed only in noncompact QED. So we use noncompact theory, because only in the noncompact QED there are lattice analogues of A_ν and $F_{\rho\sigma}$ — $\theta_{l,\nu}(x)$ and $\theta_{p,\rho\sigma}(x)$, respectively. There are two requirements for lattice representation of Wilson bag:

1) The whole integral for Wilson bag should be the gauge invariant quantity.

2) «Locality». It means that in lattice representation of the product $A_\nu F_{\rho\sigma}$, A_ν and $F_{\rho\sigma}$ should be given in the same point x. This requirement is nontrivial, because $\theta_{p,\rho\sigma}$ gives the value of $F_{\rho\sigma}$ in the center of the plaquette, but θ_ν gives the value of A_ν in the center of the link. These are different points.

These two requirements lead to the following structure of lattice representation for CS action:

$$S_{\text{CS}} = \frac{1}{8}\beta \sum_{x\in V_3} \varepsilon_{\mu\nu\rho\sigma} n_\mu(x)(\theta_{l,\nu}(x) + \theta_{l,\nu}(x+\hat\rho) + \theta_{l,\nu}(x+\hat\sigma) + \\ + \theta_{l,\nu}(x+\hat\rho+\hat\sigma))(\theta_{p,\rho\sigma}(x) + \theta_{p,\rho\sigma}(x+\hat\nu)).$$

The expression $\theta_{l,\nu}\theta_{p,\rho\sigma}$ gives us $a^3 e^2 A_\nu F_{\rho\sigma}$ in continuum limit and by means of the factor $\beta = 1/e^2$ one can eliminate e^2 from the action.

4. LATTICE DESCRIPTION OF DIELECTRIC

The same method, based on analogy to Wilson loop, can be used for lattice description of the dielectric volumes.

The simplest dielectric (when we neglect anisotropy and $\varepsilon(\omega)$ dependence) can be described by the following action:

$$S = \frac{1}{4}\int\limits_{\bar{V}} F_{\mu\nu}F^{\mu\nu}dV + \frac{1}{2}\int\limits_{V}\left(\varepsilon\sum_{i=1}^{3}F_{0i}F^{0i} + \sum_{i<j}F_{ij}F^{ij}\right)dV.$$

Here V is four-dimensional volume occupied by dielectric. Additional action in this case can be written as

$$S_{\rm add} = \frac{(\varepsilon-1)}{2}\int\limits_{V}\left(\sum_{i=1}^{3}F_{0i}F^{0i}\right)dV.$$

Its representation in noncompact QED lattice in Euclidean time is as follows:

$$S_{\rm add.lat} = \frac{\varepsilon-1}{2}\beta\sum_{x\in V}\sum_{i=1}^{3}\theta^2_{p,0i}(x).$$

And we can obtain the Casimir energy of dielectric body by calculation of the configuration average:

$$\langle {\rm e}^{iS_{\rm add.lat}}\rangle \to C\,{\rm e}^{-E_{\rm Cas}T},$$

where T is the size of 4D volume V in t direction.

5. CONTINUOUS LIMIT

The last point is the consideration of the continuous limit of our calculations.

The first feature of continuous limit procedure is that in our Casimir calculations nothing depends on β. Casimir effect without radiational corrections does not depend on the electron charge in the continuous theory. On the lattice the independence appears due to the following reasons: 1) there is no phase transition in the noncompact lattice QED, and β here only plays the role of the scale parameter for a numerical value of link variables; 2) we eliminate this dependence from observable quantities due to the multiplication by β in the final expressions for lattice actions. So β can be chosen in rather a large interval according to calculational convenience.

The second stage is «infinite lattice volume» limit $N\to\infty$. One of the basic requirements for continuous limit procedure is rotational symmetry restoration. For non-Abelian theories this restoration occurs due to large correlation length (or, in other words, small physical lattice size). So we can restore rotational symmetry by increasing correlation length. This procedure cannot be performed in noncompact Abelian but anyway rotational symmetry is restored at sufficiently large lattice distances [10]. So we do not need large correlation length for our continuous limit procedure. Instead of it we have limit $N\to\infty$, where N is the size of the lattice.

CONCLUSIONS

In this work we have proposed two new lattice variables that, by analogy to Wilson loop, describe Casimir interaction. The first of them is Wilson «bag». It is generated by Chern–Simons boundary conditions. This variable can be calculated exactly as gauge invariant quantity only in noncompact theory. The second variable is based on the simplest model of dielectric material. For both variables continuous limit procedure exists which consists of one stage: «large lattice limit».

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