Письма в ЭЧАЯ. 2009. Т. 6, № 7(156). С. 17–24

КОМПЬЮТЕРНЫЕ ТЕХНОЛОГИИ В ФИЗИКЕ

INDIVIDUAL EVENTS AND MATHEMATICAL FORMALISM OF QUANTUM MECHANICS

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We describe a scheme for constructing quantum mechanics in which the Hilbert space and linear operators are only secondary structures of the theory, while the primary structures are the elements of a noncommutative algebra (observables) and the functionals on this algebra, associated with the results of a single observation.

Описана схема построения квантовой механики, в которой гильбертово пространство и линейные операторы являются вторичными структурами теории. Первичными структурами являются элементы некоммутативной алгебры (наблюдаемые) и функционалы на этой алгебре, ассоциированные с результатами индивидуальных наблюдений.

PACS: 03.67.Hk, 01.30.Cc, 03.67.-a

The act of calculation (including quantum) or of communication is an individual event. However, there is no adequate counterpart of an individual event in a standard mathematical formalism of quantum mechanics [1]. I describe a scheme of generalization of the mathematical formalism, in which such an adequate counterpart is present.

The fundamental notion of both classical and quantum physics is an observable. In a physical system, an observable is an attribute whose numerical value can be obtained using some measuring procedure. In what follows, I assume that all observables are dimensionless, which implies fixing some system of units.

In each measurement, the investigated physical system is subject to the action from the measuring device. Therefore, all measurements can be divided into two types: reproducible and nonreproducible. A characteristic feature of reproducible measurements is then that repeated measurement of the same observable gives the originally obtained value.

The problem of reproducibility is of particular interest in the case that a number of observables are measured for the same physical system. Let us first measure observable $\hat{A}$, then observable $\hat{B}$, then again observable $\hat{A}$ and, finally, observable $\hat{B}$. If the results of the repeated measurements coincide with the original ones, I call such measurements compatible. If there are devices allowing one to make combined measurements of observables $\hat{A}$ and $\hat{B}$, I call such observables compatible or simultaneously measurable.

Experiment shows that a key difference between classical and quantum physical systems is as follows. For classical systems, the experiment can always be designed such that the

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measurements of any two observables are compatible. But a quantum system always has observables for which a compatible measurement cannot be realized in any case.

I let $\mathfrak{A}_+$ denote the set of all observables in a physical system under consideration and let $\mathfrak{Q}_\xi$ denote its maximal subset of compatible observables. It is clear that for a classical system, this subset coincides with the set $\mathfrak{A}_+$ itself. For a quantum system, it can be verified that they are infinitely many [2]. The index ξ, ranging a set Ξ, distinguishes one such subset from another. A given observable can belong to different subsets $\mathfrak{Q}_\xi$ simultaneously.

It is easy to verify that each subset $\mathfrak{Q}_\xi$ can be endowed with the structure of a real commutative associative algebra. Indeed, experiment shows that for any compatible observables $\hat{A}$ and $\hat{B}$, there exists a third observable $\hat{D}$ such that, first, it is compatible with the observables $\hat{A}$ and $\hat{B}$, and, second, in each simultaneous (compatible) measurement of these three observables, the measurement results are related by

$$AB = D.$$

Because this relation is satisfied independently of the values of the individual constituents, it can be assumed that the observables themselves by definition satisfy the same relation:

$$\hat{A}\hat{B} = \hat{D}.$$

We can similarly define the addition operation for two compatible observables and the operation of multiplication of an observable by a real number.

For a given physical system, using compatible measurements, we assign to each observable $\hat{A} \in \mathfrak{Q}_\xi$ a measurement result:

$$\hat{A} \to A = \varphi_\xi(\hat{A}).$$

This defines a functional $\varphi_\xi(\hat{A})$ on the algebra $\mathfrak{Q}_\xi$. By the definition of the algebraic operations in $\mathfrak{Q}_\xi$, this functional is a homomorphic map of $\mathfrak{Q}_\xi$ into the set of real numbers.

Such a functional is called a character of a real commutative associative algebra (see, e.g., [3]).

These characters possess a lot of specific properties, which allow formulating the recipe of their construction [2].

The mathematical representation of a physical system is the set of observables of this system. In what follows, I identify the physical system with the set of its observables in which relations among observables are fixed. I identify a subset of observables with the corresponding physical subsystem. In doing this, I do not assume that the subsystem must necessarily be somehow isolated from the rest of the system. The subsystem may not be isolated spatially and can interact with other parts.

Next discuss the notion of a state of a physical system. First consider a classical system. In this case, a state of the physical system is understood as its attribute that uniquely predetermines the results of measurements of all the observables. Mathematically, a state is usually given by a point in phase space. But it can be easily reckoned that this is just one specific version of fixing a certain functional on the algebra of observables, which is a character of this algebra.

To be free from the choice of any specific version, I define a state of a classical system as a character of the algebra of observables of this system.

This definition extends to a quantum system as follows.

I consider the set $\mathfrak{A}_+$ of observables of a quantum system as a collection of subsets $\mathfrak{Q}_\xi$ ($\xi \in \Xi$), each of which is a maximal subset of compatible observables. Each of these subsets has the structure of a real commutative associative algebra and can be considered the algebra of observables of some classical subsystem of the quantum system. These classical subsystems are open, but we can still describe the state of each of them using a character $\varphi_\xi(\cdot)$ of the corresponding algebra $\mathfrak{Q}_\xi$.

I say that an elementary state of a physical system is a collection $\varphi = [\varphi_\xi]$ ($\xi \in \Xi$) of functionals $\varphi_\xi(\cdot)$ each of which is a character of the corresponding algebra $\mathfrak{Q}_\xi$.

In each individual measurement, we can measure the observables belonging to any given $\mathfrak{Q}_\xi$ algebra. The results of such a measurement are defined by the corresponding functional $\varphi_\xi(\cdot)$ belonging to the collection φ.

Thus, the result of each individual measurement of observables of a physical system is determined by the elementary state of this system.

This statement holds for both classical systems (in which case the collection $\varphi = [\varphi_\xi]$ consists of a single functional) and quantum systems (in which case the collection $\varphi = [\varphi_\xi]$ consists of infinitely many functionals).

I note that I do not assume the validity of the equality

$$\varphi_\xi(\hat{A}) = \varphi_{\xi'}(\hat{A}), \text{ if } \hat{A} \in \mathfrak{Q}_\xi \cap \mathfrak{Q}_{\xi'}. \tag{1}$$

Although deceptively natural, this assumption has no experimental justification (see [2]).

That condition (1) is not satisfied implies that a measurement result can depend not only on the system investigated (on its elementary state) but also on the type of device used for measurement.

I say that measuring devices belong to the ξ type if for a system in the elementary state $\varphi = [\varphi_\xi]$, the measurement result for each observable $\hat{A} \in \mathfrak{Q}_\xi$ is described by the functional $\varphi_\xi(\cdot)$. Obviously, condition (1) may be satisfied for some φ. If (1) holds for all the $\mathfrak{Q}_\xi$ containing $\hat{A}$, then we say that the elementary state $\varphi = [\varphi_\xi]$ is stable for the observable $\hat{A}$.

An elementary state of a quantum system cannot be uniquely fixed experimentally because the most that can be measured in a single experiment is the observables belonging to one algebra $\mathfrak{Q}_\xi$. As a result, only values of the functional $\varphi_\xi(\cdot)$ can be determined. The elementary state φ remains otherwise undefined.

For determining the values of other observables, an additional experiment must be performed involving a device incompatible with the one used previously. The new device uncontrollably perturbs the elementary state that had occurred after the first measurement. Therefore, the information obtained in the first experiment becomes obsolete.

In view of this, it is convenient to unite the elementary states φ having the same restriction to the algebra $\mathfrak{Q}_\xi$ (i.e., the functional $\varphi_\xi(\cdot)$) into a class φ_ξ-equivalent elementary states. Thus, only the equivalence class to which the elementary state of the considered system belongs can be established in a quantum measurement.

A reproducible measurement of observables belonging to an algebra $\mathfrak{Q}_\xi$ is reminiscent of the procedure for preparing a quantum state in the standard quantum mechanics. Accordingly, the class $\{\varphi\}_{\varphi_\xi}$ of φ_ξ-equivalent elementary states φ that are stable on the subalgebra $\mathfrak{Q}_\xi$ is said to be a quantum state Ψ_{φ_ξ}.

Defying the widely shared opinion that the Kolmogorov probability theory [4] is inapplicable to quantum systems, I try to use it.

The fundamental notion of the Kolmogorov probability theory is the probability space [4, 5]). This is a triple $(\Omega, \mathcal{F}, P)$. The first term in the triple, Ω, is a set (space) of elementary events.

The defining properties of elementary events are as follows: (a) one and only one elementary event occurs in each trial; (b) elementary events exclude each other. Because two nonorthogonal quantum states do not exclude each other, they cannot be elementary events. In our case, the role of an elementary event is played by the elementary state φ.

In addition to the elementary event, the notion of an event is also introduced. Each event F is identified with some subset of the set Ω. An event F is considered to have occurred if one of the elementary events belonging to this subset ($\varphi \in F$) occurred. The collections of subsets F of the set Ω are endowed with the structure of a Boolean algebra.

I recall that the Boolean algebra of a set Ω is the system of subsets of this set with three algebraic operations defined on it: taking the union of subsets, the intersection of subsets, and the complement of each subset in Ω. Accordingly, the second term in the triple is some Boolean algebra $\mathcal{F}$.

Finally, the third term in the triple is a probability measure P. This is a map of $\mathcal{F}$ into the set of real numbers $P(F)$ satisfying the conditions (a) $0 \leqslant P(F) \leqslant 1$ for all $F \in \mathcal{F}$, $P(\Omega) = 1$ and (b) $P(\sum_j F_j) = \sum_j P(F_j)$ for any denumerable collection of nonintersecting subsets $F_j \in \mathcal{F}$. The probability measure is defined only for the events from the algebra $\mathcal{F}$. For elementary events, the probability may not exist in general.

From the physical standpoint, the choice of a Boolean algebra $\mathcal{F}$ is determined by the characteristics of the measuring devices used. The point is that in reality, measuring devices have a finite resolving power and therefore cannot always differentiate one elementary event from another. They can then only be used to establish that a given experiment involves one of the elementary events belonging to some subset.

Here is the key difference between classical and quantum physical systems. In the classical case, we can infinitely increase the resolving power and use devices that allow simultaneously measuring the values of an arbitrary number of observables. In the quantum case, compatible measurements can be performed only for observables belonging to a given algebra $\mathfrak{Q}_\xi$. Such measurements correspond to a certain type of Boolean algebra, denoted by $\mathcal{F}_\xi$ in what follows.

The elements of this Boolean algebra differ in the values (intervals of values) of observables in the algebra $\mathfrak{Q}_\xi$. More detailed measurements in which the values of observables not belonging to $\mathfrak{Q}_\xi$ are additionally measured are not allowed, because they are incompatible with the previous measurements. Therefore, the Boolean algebras whose elements additionally differ in the values of observables not belonging to $\mathfrak{Q}_\xi$ are useless. No probability measure corresponds to such Boolean algebras.

The choice of some Boolean algebra $\mathcal{F}_\xi$, mathematically speaking, makes the set Ω of elementary events into a measurable space $(\Omega, \mathcal{F}_\xi)$. In an experiment, this space corresponds to a pair: the physical object under investigation and a certain type (type ξ) of measuring device allowing compatible measurements of observables from the algebra $\mathfrak{Q}_\xi$.

I say that a quantum ensemble is a set of physical systems that are described by the same set $\mathfrak{A}_+$ of observables and are in some quantum state. Experiment shows that a quantum ensemble has probabilistic properties. It must therefore admit the introduction of a probability space structure. As a result of a reproducible measurement, the quantum ensemble passes into a new quantum ensemble with another probability distribution.

I consider the quantum ensemble of systems that are in a quantum state Ψ_{φ_η} $(\eta \in \Xi)$. The space $\Omega(\varphi_\eta)$ of elementary events for this ensemble is given by the equivalence class $\{\varphi\}_{\varphi_\eta}$. Let a type-$\xi$ device be used in the experiment. This corresponds to a measurable space $(\Omega(\varphi_\eta), \mathcal{F}_\xi)$ and a probability measure P_ξ.

I measure an observable $\hat{A} \in \mathfrak{Q}_\xi$ and say the event F_A occurs in the experiment if the registered value of $\hat{A}$ is not greater than A. Let $P_\xi(A) = P(\varphi : \varphi_\xi(\hat{A}) \leqslant A)$ denote the probability of this event. If the observable $\hat{A}$ also belongs to an algebra $\mathfrak{Q}_{\xi'}$, then a ξ'-type device could be used to determining the probability of F_A. In this case, a different value $P_{\xi'}(A)$ could be obtained for the probability. But experiment shows that the same probability is obtained in this case, i.e.,

$$P(\varphi : \varphi_\xi(\hat{A})) \leqslant A) = P(\varphi : \varphi_{\xi'}(\hat{A}) \leqslant A). \tag{2}$$

I introduce the notation

$$P_{\hat{A}}(d\varphi) = P(\varphi : \varphi(\hat{A}) \leqslant A + dA) - P(\varphi : \varphi(\hat{A}) \leqslant A),$$

where the subscript ξ on the functional $\varphi(\hat{A})$ is omitted in view of (2).

To find the mean of an observable $\hat{A}$ in a quantum state Ψ_{φ_η}, I need not consider observables that are incompatible with $\hat{A}$. Therefore, instead of considering the quantum system, I can restrict myself to considering its classical subsystem whose observables are described by the algebra $\mathfrak{Q}_\xi$ $(\hat{A} \in \mathfrak{Q}_\xi)$. To determine the mean $\langle \hat{A} \rangle$, I can then use the mathematical formalism of classical probability theory (see, e.g., [5]) and write

$$\langle \hat{A} \rangle = \int_{\varphi \in \Psi_{\varphi_\eta}} P_{\hat{A}}(d\varphi)\, \varphi(\hat{A}) \equiv \Psi_{\varphi_\eta}(\hat{A}). \tag{3}$$

Formula (3) defines a functional $\Psi_{\varphi_\eta}(\hat{A})$ (a quantum mean) on the set $\mathfrak{A}_+$.

Obviously, $\Psi_{\varphi_\eta}(\alpha\hat{A}) = \alpha\Psi_{\varphi_\eta}(\hat{A})$, where α is any real number.

Experiment also shows that for any $\hat{A} \in \mathfrak{A}_+$ and $\hat{B} \in \mathfrak{A}_+$, there exists an observable $\hat{D} \in \mathfrak{A}_+$ such that the relation

$$\Psi\varphi_\eta(\hat{A}) + \Psi\varphi_\eta(\hat{B}) = \Psi\varphi_\eta(\hat{D})$$

holds for each quantum state $\Psi\varphi_\eta(\cdot)$. Such an element $\hat{D}$ can by definition be considered the sum of $\hat{A}$ and $\hat{B}$. This means that the set $\mathfrak{A}_+$ can be endowed with the structure of a real linear space such that the $\Psi\varphi_\eta(\cdot)$ are linear functionals on this space.

Due to properties of characters $\varphi_\xi(\cdot)$ these functionals are positive.

Because any observable $\hat{A} \in \mathfrak{A}_+$ is compatible with itself, it follows that the operation of taking the square of $\hat{A}$ can be defined on the set $\mathfrak{A}_+$ following the same scheme as $\mathfrak{Q}_\xi$.

This allows endowing the linear space $\mathfrak{A}_+$ with the structure of a real Jordan algebra [6,7] with the product of elements $\hat{A}$ and $\hat{B}$ defined as

$$\hat{A} \circ \hat{B} = \frac{1}{2}\left((\hat{A} + \hat{B})^2 - \hat{A}^2 - \hat{B}^2\right). \tag{4}$$

This product is manifestly commutative but not associative in general.

All the Jordan algebras are divided into two classes: special and exceptional. Whether exceptional Jordan algebras can be used in quantum physics is unknown. In all quantum models considered to date, the set of observables can be endowed with the structure of a special Jordan algebra. In line with this historical experience, we shall be limited to special algebras of Jordan.

A Jordan algebra is special if there exists an associative (not necessarily real and commutative) algebra $\mathfrak{A}$ such that the set $\mathfrak{A}_+$ as a linear space is a subspace in $\mathfrak{A}$. Therefore, we shall assume the set $\mathfrak{A}_+$ of observables coincides with the set of Hermitian elements of the involutive algebra $\mathfrak{A}$.

The elements of the algebra $\mathfrak{A}$ are called dynamical variables in what follows. Any element $\hat{U} \in \mathfrak{A}$ is uniquely represented as $\hat{U} = \hat{A} + i\hat{B}$, where $\hat{A}, \hat{B} \in \mathfrak{A}_+$. Therefore, the functional $\Psi_{\varphi_\eta}(\,\cdot\,)$ can be uniquely extended to a linear functional on $\mathfrak{A}$ as $\Psi_{\varphi_\eta}(\hat{U}) = \Psi_{\varphi_\eta}(\hat{A}) + i\Psi_{\varphi_\eta}(\hat{B})$.

It is possible to show (see [2]) that the equality

$$\sup_{\eta}\sup_{\varphi_\eta} \Psi_{\varphi_\eta}(\hat{U}^*\hat{U}) = \sup_{\eta}\sup_{\varphi_\eta} \varphi_\eta(\hat{U}^*\hat{U})$$

holds; therefore (see [2, 7]), a norm can be introduced in the algebra $\mathfrak{A}$ using the equality

$$\|\hat{U}\|^2 = \sup_{\eta}\,\sup_{\varphi_\eta} \varphi_\eta(\hat{U}^*\hat{U}).$$

Because the functional φ_η is a character of $\mathfrak{Q}_\eta$, we have $\varphi_\eta([\hat{U}^*\hat{U}]^2) = [\varphi_\eta(\hat{U}^*\hat{U})]^2$. This implies that

$$\|\hat{U}^*\hat{U}\| = \|\hat{U}\|^2. \tag{5}$$

A complete normalized involutive algebra whose norm satisfies additional condition (5) is called a C^* algebra [3]. Therefore, the algebra of quantum dynamical variables can be endowed with the structure of a C^* algebra.

A remarkable property of C^* algebras is that any C^* algebra is isomorphic to a subalgebra of linear bounded operators in an appropriate Hilbert space $\mathfrak{H}$ [3]. A faithful representation of the C^* algebra is said to be realized in the space $\mathfrak{H}$. The relation of a C^* algebra to a Hilbert space is realized by the so-called canonical Gelfand–Neimark–Segal (GNS) construction (see, e.g., [7, 8]).

It is easy to show that average value of an observable $\hat{A}$, which is defined by the equation

$$\langle\hat{A}\rangle = \int\limits_{\varphi\in\Psi_{\varphi_\eta}} P_{\hat{A}}(d\varphi)\,\varphi(\hat{A}) \equiv \Psi_{\varphi_\eta}(\hat{A}),$$

can be represented in the form

$$\langle\Phi|\Pi(\hat{A})|\Phi\rangle. \tag{6}$$

Here $|\Phi\rangle$ is a vector in the Hilbert space $\mathfrak{H}$ constructed by GNS, and $\Pi(\hat{A})$ is the operator that corresponds to an observable $\hat{A}$ in space $\mathfrak{H}$. This allows the full use of the mathematical formalism of the standard quantum mechanics in calculating quantum means in the proposed approach.

At the same time, the proposed approach differs essentially from the standard quantum mechanics. In the latter, a relation of type (6) is postulated (Born's postulate [9]) and is the starting point for constructing the so-called quantum probability theory. But unlike the classical probability theory, the quantum probability theory is not yet constructed as a nice mathematical scheme.

Here, formula (6) is derived as a consequence of physically justified statements and of the classical probability theory. In addition, we indicate when the formula is valid: Eq. (6) is applicable for calculating means of observables over a quantum ensemble.

The approach I have presented by no means rejects standard quantum mechanics. The founding fathers of quantum mechanics erected a remarkable edifice. But they began the construction with the second floor, the description of probabilities and means.

Therefore, the stability of this edifice has required a large amount of shoring in the form of a series of «principles»: the superposition principle, the uncertainty principle, the principle of complementarity, the projection principle, the indistinguishability principle, and the principle of the absence of trajectories.

All these principles appear rather artificial and are not easily amenable to physical interpretation. The main task of these principles is to justify the mathematical formalism of the standard quantum mechanics. True, this mathematical formalism has proved amazingly serendipitous, but this is not the case with its physical interpretation.

It is not without reason that discussions of the physical interpretation of quantum mechanics still vividly proceed, although the term «physical interpretation» itself seems quite strange. If quantum mechanics is a physical theory, then it must not need any physical interpretation. By using this term, we admit, be it willingly or not, that quantum mechanics is not a physical theory but a mathematical model.

In this work, I have attempted to construct quantum mechanics just as a physical theory, based on experimental data.

The central point of the described approach is the introduction of the notion of an «elementary state», which is absent in the formalism of the standard quantum mechanics. This notion, on the one hand, gives a clear mathematical counterpart of such a physical phenomenon as an individual experimental act.

On the other hand, it allows using the well-developed formalism of classical logic and classical probability theory. It must be borne in mind here that although the references to the so-called quantum logic and quantum probability theory may be rather frequent, it has so far been impossible to give them the structure of a clear-cut complete theoretical scheme.

Based on the notion of an elementary state and using the *classical* probability theory, we can completely reproduce the mathematical formalism of the standard quantum mechanics and simultaneously show its applicability domain. This formalism applies to quantum ensembles. This is a very important type of ensemble but not the most general one by far. In particular, this formalism is not suitable for describing an individual event.

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