BPS domain wall in massive nonlinear sigma model in harmonic superspace

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BPS wall solutions in four–dimensional massive $\mathcal{N}=2$ nonlinear sigma models are studied in the off-shell harmonic superspace approach in which $\mathcal{N}=2$ supersymmetry is manifest. The general nonlinear sigma model can be described by an analytic harmonic potential which is the hyper–Kähler analog of the Kähler potential in $\mathcal{N}=1$ theory. We examine the massive nonlinear sigma model with multi–center four–dimensional target hyper–Kähler metrics and derive the corresponding BPS equation. We study in some detail two particular cases with the Taub–NUT and double Taub–NUT metrics. The latter embodies, as its two separate limits, both Taub–NUT and Eguchi–Hanson metrics. We find that domain wall solutions exist only in the double Taub–NUT case including its Eguchi–Hanson limit.

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1 Introduction

It is well known that topological solutions are of importance in various areas of particle physics. Recently, there was renewed interest in such solutions because of their crucial role in the brane world scenario [1, 2, 3]. In this scenario, our world is assumed to be realized on topological objects like domain walls or brane–junctions. Investigating the quantum fluctuation of the domain wall, it was found that zero modes are localized on the wall [4] and the low energy theory becomes a theory on the wall. In other words, domain wall background gives rise to some kind of the dimensional reduction as an alternative to the standard Kaluza–Klein compactification [5]. Supersymmetry (SUSY) can also be implemented in these models, and it is actually a powerful device for constructing their topological solutions. In SUSY theories, these often appear as the BPS states [6] which spontaneously break a part of the original SUSY [7]. Viewing the four–dimensional world as a domain wall, we are led to deal with SUSY theories in five dimensions. The minimal possibility is $\mathcal{N}=1$, $d=5$ SUSY possessing eight supercharges.

SUSY with eight supercharges is very restrictive. For instance, in theories involving only massless scalar multiplets (hypermultiplets), non-trivial interactions can only arise from nonlinearities in kinetic terms. Prior to studying the genuine five–dimensional theories with hypermultiplets, it is instructive to start with similar SUSY theories in four dimensions, i.e., $\mathcal{N}=2$, $d=4$ theories. Actually, in

$\mathcal{N}=1$, $d=5$ and $\mathcal{N}=2$, $d=4$ theories, the hypermultiplets contain the same number of on-shell components, viz., two complex scalars and one Dirac fermion. This analysis of the four–dimensional theory could then be of help in studying the brane world scenarios based on SUSY theories in higher dimensions [8, 9, 10].

With regard to rigid $\mathcal{N}=2$ SUSY the target manifold of the hypermultiplet $d=4$ sigma models must be hyper–Kähler (HK) [11]. In these theories, the scalar potential can be obtained only if the hypermultiplets acquire masses by the Scherk–Schwarz mechanism [12] because of the appearance of central charges in the $\mathcal{N}=2$ Poincaré superalgebra [13]. The form of the potential is specified by the norm of the Killing vector of the target manifold isometry whose generator is identified with the central charge [14]. We call a nonlinear sigma model (NLSM) on the HK target manifold with mass term *massive* nonlinear HK sigma model. Various topological solutions in the model have been studied in terms of not only on-shell framework [15, 14, 16, 17, 18, 19] but also off-shell formalism [20, 21, 22].

An off-shell formalism is appropriate for such a study since it provides a powerful tool of constructing models with the domain wall and brane–junction solutions [23], as well as a low–energy effective action around the wall [24]. The most natural description of $\mathcal{N}=2$, $d=4$ SUSY field theories is achieved in the harmonic superspace (HSS) [25, 26]. The HSS approach is the only one to allow superfield formulations of $\mathcal{N}=2$ SUSY theories with all supersymmetries being manifest and off-shell. In the HSS approach, any HK nonlinear sigma model can be described by one analytic function which is the HK analog of Kähler potential. This analytic function (HK potential) embodies self-interactions of hypermultiplets.

The purpose of our work is to investigate $\mathcal{N}=2$ *massive* nonlinear sigma models in the HSS approach. We limit ourselves to the case of sigma models associated with four–dimensional HK multi–center metrics, because everything is drastically simplified in this case. The component action (both its kinetic and potential parts) can be written in terms of the single analytic HK potential. The resulting scalar component potential turns out to coincide with that in [14]. The general form of the BPS equation is derived in the multi–center case. As examples we consider sigma models associated with the Taub–NUT [27] metric and its generalization, the so called double Taub–NUT (DTN) metric (see e.g. [28]). The latter encompasses both the Taub–NUT and Eguchi–Hanson [29] metrics as its two limiting cases. We demonstrate that only in the double Taub–NUT and Eguchi–Hanson cases BPS domain wall solutions exist. The condition of the existence of SUSY vacua comes out as some restriction on the analytic HK potential, similarly to the $\mathcal{N}=1$ case where there arise analogous restrictions on the superpotential and Kähler potential. This criterion might be useful in constructing other $\mathcal{N}=2$ models with domain wall solutions.

This paper is based on our original paper [30]. [1])

[1]) We follow the notation: $\mathrm{diag}(\eta_{\mu\nu})=(1,-1,-1,-1)$ and $\epsilon^{12}=-\epsilon_{12}=-1$.

2 General massive nonlinear sigma model in the harmonic superspace

First, we study the general massive nonlinear sigma model with at least one triholomorphic (i.e., commuting with supersymmetry) $U(1)$ isometry. The presence of such an isometry is necessary if one wishes to gain the mass (and/or scalar potential) terms. We shall not specify how this isometry is realized. Next we examine the particular case of the four–dimensional target HK space. In this case, requiring the theory to have an $U(1)$ isometry implies that the corresponding HK metric falls into the multi–center class [28]. As was shown in [28], using some coordinate transformation, this $U(1)$ isometry can always be cast in the form in which it is realized as some phase or purely shift transformation of the coordinates of the HK manifold. For this case we shall demonstrate that the scalar potential is given by the square of the isometry Killing vector, in accord with the result of [14].

2.1 HK sigma model in HSS: the general massless case

First, we consider the action of the general massless nonlinear sigma model in the HSS approach.

The HSS action for a general nonlinear $\mathcal{N}=2$, $d=4$ supersymmetric sigma model which yields in the bosonic sector a sigma model with $4n$ dimensional HK target space is just the general superfield action of n hypermultiplets. In the HSS formalism, the hypermultiplet is described by an analytic superfield q^+_a ($a=1,\dots,n$ is a flavor index of fundamental representation of $Sp(n)$) which is a function given on the harmonic analytic $\mathcal{N}=2$ superspace

$$\{\zeta_A, u^\pm_i\} \equiv \{x^\mu_A = x^\mu - 2i\theta^{(i}\sigma^\mu\bar\theta^{j)}u^+_{(i}u^-_{j)},\ \theta^+ = \theta^i u^+_i,\ \bar\theta^+ = \bar\theta^i u^+_i,\ u^\pm_i\}\,, \tag{1}$$

where the coordinates u^{+i}, u^{-i}, $u^{+i}u^-_i = 1$, $i=1,2$ [2]) are the $SU(2)_R/U(1)$ harmonic variables [25, 26].

Exploiting the target space reparameterization covariance, the general action can be cast in the form [26]

$$S = \tfrac{1}{2}\int \mathrm{d}\zeta^{(-4)}_A \mathrm{d}u\,\left[q^+_a D^{++} q^{a+} + L^{+4}(q^+_a, u^\pm_i)\right]\,, \tag{2}$$

where $\mathrm{d}\zeta^{(-4)}_A\mathrm{d}u = \mathrm{d}^4x_A\mathrm{d}^2\theta^+\mathrm{d}^2\bar\theta^+\mathrm{d}u$ is the measure of integration over analytic superspace (1), D^{++} is the harmonic covariant derivative defined as

$$D^{++} = \partial^{++} - 2\mathrm{i}\theta^+\sigma^\mu\bar\theta^+\partial_\mu\,, \quad \partial^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} \tag{3}$$

and $L^{+4}(q^+_a, u^\pm_i)$ is the analytic HK potential. The analytic superfield q^+_a can be expanded as

$$\begin{aligned} q^+_a(\zeta_A, u^\pm_i) = {} & F^+_a + \sqrt{2}\theta^+\psi_a + \sqrt{2}\bar\theta^+\bar\varphi_a + \mathrm{i}\theta^+\sigma^\mu\bar\theta^+ A^-_{a\mu} + \theta^+\theta^+ M^-_a + \bar\theta^+\bar\theta^+ N^-_a \\ & + \sqrt{2}\theta^+\theta^+\bar\theta^+\bar\chi^{--}_a + \sqrt{2}\bar\theta^+\bar\theta^+\theta^+\xi^{--}_a + \theta^+\theta^+\bar\theta^+\bar\theta^+ P^{(-3)}_a\,, \end{aligned} \tag{4}$$

[2]) In what follows, $a,\dots,f$ stand for the $Sp(n)$ indices and $i,j,\dots$ for the $SU(2)_R$ indices, respectively.

and it satisfies the reality condition

$$\widetilde{q_a^+} = \Omega^{ab} q_b^+ \,. \tag{5}$$

Here Ω^{ab} is the skew-symmetric constant $Sp(n)$ metric, $\Omega^{ab}\Omega_{bc} = \delta^a_c$, and "$\sim$" denotes a (pseudo) conjugation which is the product of the complex conjugation (denoted by "$-$") and the star (pseudo) conjugation [26]. The action of the "$\sim$" conjugation on ζ_A and $u_i^\pm$ is defined as

$$\widetilde{x_A^\mu} = x_A^\mu \,, \quad \widetilde{\theta^+} = \bar\theta^+ \,, \quad \widetilde{\bar\theta^+} = -\theta^+ \,, \quad \widetilde{u_i^\pm} = u^{\pm i} \,, \quad \widetilde{u^{\pm i}} = -u_i^\pm \,. \tag{6}$$

In what follows, we shall frequently omit the $Sp(n)$, $SU(2)_R$ and space–time indices of the arguments in the analytic functions, e.g. write $f = f(q_a^+, u_i^\pm) = f(q,u)$.

The action (2) is assumed to be invariant under the following isometry transformation [3])

$$\delta q^{a+} = \varepsilon\, \lambda^{a+}(q,u) \,, \tag{7}$$

provided that $\lambda^{a+}(q,u)$ satisfies the equations ($\partial_{a+} = \partial/\partial q^{a+}$)

$$\lambda^{a+} = \tfrac{1}{2}\, \Omega^{ab} \partial_{b+} \Lambda^{++} \,, \tag{8}$$

$$\partial^{++}\Lambda^{++} - \tfrac{1}{2}\, \Omega^{ab} \partial_{a+} L^{+4} \partial_{b+} \Lambda^{++} = 0 \,. \tag{9}$$

In eq. (7), ε is a group parameter. The quantities λ^{a+} and Λ^{++} are referred to as the superfield Killing vector and Killing potential, respectively. In what follows we shall need eqs. (7)$-$(9) only in the limit when all fermions are discarded, which amounts to the reduction $q^+ \to F^+$.

From now on, we neglect all fermionic fields and deal with the bosonic component action. Both fermionic and bosonic components in (4) contain infinite sets of auxiliary fields coming from the harmonic expansions. In order to obtain the action in terms of $4n$ physical bosonic fields only, we should eliminate the relevant auxiliary fields by solving their algebraic (i.e., kinematical) equations of motion. Therefore, as the basic steps towards the final sigma model action we should single out the kinematical part of the equations of motion following from (2) (with all fermions being discarded) and solve these equations.

Substituting the bosonic part of Grassmann expansion (4) into the action (2), and integrating over Grassmann coordinates, we obtain the bosonic action in the form

$$\begin{aligned} S_{\rm bos} = \int \mathrm{d}^4x_A \mathrm{d}u \Big\{ & \tfrac{1}{4}\, A_\mu^{a-} \left[\left(D^{++}\delta_a{}^b - \tfrac{1}{2}\, \partial_{a+}\partial_{c+}\Omega^{cb} L^{+4} \right) A_b^{-\mu} - 4\partial^\mu F_a^+ \right] \\ & - M^{a-} \left(D^{++}\delta_a{}^b - \tfrac{1}{2}\, \partial_{a+}\partial_{c+}\Omega^{cb} L^{+4} \right) N_b^- \\ & \quad - P^{a(-3)} \left(D^{++} F_a^+ - \tfrac{1}{2}\, \partial_{a+} L^{+4} \right) \Big\}. \end{aligned} \tag{10}$$

[3]) In general, (2) is not obliged to respect any extra symmetry except for $\mathcal{N} = 2$ SUSY.

Here $L^{+4} = L^{+4}(F,u) \equiv L^{+4}(q,u)$ at $\theta = 0$. Equation of motion for F_a^+ is

$$D^{++}F_a^+(x,u) - \tfrac{1}{2}\,\partial_{a+}L^{+4}(F,u) = 0\,. \tag{11}$$

Here D^{++} coincides with a partial harmonic derivative ∂^{++} which acts on the harmonic arguments of the component fields as well as on the harmonics appearing explicitly. We denote the harmonic derivative in (11) by D^{++} in order to distinguish it, e.g., from the partial derivative in (9) which acts only on the explicit harmonics in $\Lambda^{++}(F,u)$ and not on the harmonic arguments of $F = F(x,u)$. We reserve the notation ∂^{++} just for this latter derivative.

The derivative D^{++}, when applied to an arbitrary scalar function $G(F,u)$, yields:

$$D^{++}G = \partial^{++}G + D^{++}F^{a+}\partial_{a+}G = \partial^{++}G + \tfrac{1}{2}\,\Omega^{ab}\partial_{b+}L^{+4}\partial_{a+}G\,, \tag{12}$$

where we used eq. (11). Defining

$$\mathcal{D}^{++}G_a = D^{++}G_a - \tfrac{1}{2}\,\partial_{a+}\partial_{b+}L^{+4}G^b\,, \tag{13}$$

equations of motion for $A^-_{a\mu}$, M_a^- and N_a^- can be rewritten as

$$\mathcal{D}^{++}A^-_{a\mu} - 2\partial_\mu F_a^+ = 0\,, \tag{14}$$

$$\mathcal{D}^{++}M_a^- = \mathcal{D}^{++}N_a^- = 0\,. \tag{15}$$

The remaining equation, which comes from the variation with respect to F_a^+, is dynamical, and we will not use it in the following (it can be reproduced in the end by varying the eventual action with respect to the dynamical fields $f^{ai}(x)$ to be defined below).

After substituting eqs. (11), (14) and (15) back into (10), the action is drastically simplified

$$S_{\rm bos} = \int \mathrm{d}^4x_A \mathrm{d}u \left(-\tfrac{1}{2}\,A^{a-}_\mu\partial^\mu F_a^+\right). \tag{16}$$

Note that the harmonic fields F_a^+ and $A^-_{a\mu}$ are still subject to the constraints (11) and (14) and they include infinite sets of auxiliary fields. Solving eq. (11), one can express $F_a^+(x_A,u)$ as $F_a^+ = F_a^+(f^{ai},u)$ where $f^{ai}(x)$ are the standard HK target space coordinates. We shall refer to

$$f^{a\pm} = f^{ai}(x)u_i^\pm \tag{17}$$

as the "central basis" HK coordinates and to F^{a+} and F^{a-} related by

$$D^{++}F^{a-} = F^{a+} \tag{18}$$

as the "analytic basis" HK coordinates [31, 26]. A more detailed explanation of this nomenclature can be found in [31, 26]. Given the solution $F_a^+ = F_a^+(f^{ai},u)$, the field $A^-_{a\mu}$ can be expressed from (14) as $A^-_{a\mu} = A^-_{a\mu}(f^{ai},u)$. Substituting these solutions into the action results in the final sigma model action for $f^{ai}(x)$.

In solving eqs. (11), (14) and (15), it is convenient to make use of the one-to-one correspondence between the HK sigma models and the geometric construction of HK manifold in the harmonic space [31, 26]. In the latter formulation, the standard constraints of the HK geometry are interpreted as the conditions of harmonic analyticity. This allows one to solve defining constraints of the HK geometry in terms of two unconstrained analytic potentials one of which proves to be pure gauge. The remaining potential encodes all the information about the given HK manifold, in the sense that all the relevant geometric objects, i.e., connections, vielbeins and metric, can be expressed in terms of this potential. We call this geometric approach the non-Lagrangian one, in contrast to the Lagrangian approach to which we adhere in this paper and in which the metric and other geometric quantities of HK geometry appear in the $\mathcal{N}=2$, $d=4$ supersymmetric sigma model type action as the result of solving equations of motion for an infinite tower of auxiliary fields contained in q^{a+}. As was shown in [31, 26], both approaches are in fact equivalent to each other. In particular, the unconstrained potential in the non-Lagrangian approach corresponds to the analytic HK potential in $\mathcal{N}=2$ nonlinear sigma model. Using the one-to-one correspondence between these two approaches, as well as the differential geometry techniques of Refs. [31, 26], it is easy to check that the solution of eqs. (14) and (15) is

$$A_\mu^{a-} = 2E_{bi}^{a-}\partial_\mu f^{bi}\,, \tag{19}$$

$$M_a^- = N_a^- = 0\,, \tag{20}$$

where E_{bi}^{a-} is one of the two central basis vielbeins $E_{bi}^{a\pm}$ from which the HK metric is constructed. They satisfy the relations [26]

$$\mathcal{D}^{++}E_{bi}^{a+} = 0\,, \quad E_{bi}^{a+} = -\mathcal{D}^{++}E_{bi}^{a-} = -\partial_{bi}F^{a+}\,, \tag{21}$$

which can be deduced based upon eq. (11) (for instance, the first one is proved by applying $\partial/\partial f^{bi}$ to (11)). Substituting (19) and (20) into (16), we obtain

$$S_{\rm bos} = \tfrac{1}{2}\int \mathrm{d}^4x_A\, g_{ai,bj}(x_A)\,\partial^\mu f^{ai}\partial_\mu f^{bj}\,, \tag{22}$$

where $g_{ai,bj}$ is the HK target space metric defined by

$$g_{ai,bj} = \Omega_{cd}(E_{ai}^{c-}E_{bj}^{d+} - E_{ai}^{c+}E_{bj}^{d-})\,. \tag{23}$$

It is easy to show that this metric is u independent, $g_{ai,bj} = g_{ai,bj}(x_A)$. This can be checked utilizing eq. (21), keeping in mind that the $Sp(n)$ connections drop out altogether due to the contraction of the $Sp(n)$ indices.

2.2 General massive HK sigma model in HSS

Next we consider the general massive deformation of the HSS q^+ Lagrangian. Suppose we are given a q^+ action possessing an isometry. Then we assign to q^+ a dependence on the central charge coordinate x^5, such that $\partial/\partial x^5$ can be identified with the Killing vector of the isometry

$$\frac{\partial}{\partial x^5}\, q^{a+} = m\, \lambda^{a+}(q,u)\,, \tag{24}$$

where m is a mass parameter which, for simplicity, is taken to be real. Correspondingly, the harmonic covariant derivative (3) acquires the central charge term [32]:

$$D^{++} \;\to\; D^{++} + \mathrm{i}\left[(\theta^+)^2 - (\bar\theta^+)^2\right]\frac{\partial}{\partial x^5}\,, \tag{25}$$

and the action (2) is modified as

$$S = \tfrac{1}{2}\int \mathrm{d}\zeta_A^{(-4)}\mathrm{d}u\big(q_a^+ D^{++} q^{a+} + L^{+4}(q,u) + \mathrm{i}m\left[(\theta^+)^2 - (\bar\theta^+)^2\right] q_a^+\lambda^{a+}\big)\,. \tag{26}$$

Similarly to the massless case, substituting the Grassmann expansion of the harmonic superfield (4) with suppressed fermions into (26), we obtain the bosonic action in the following form

$$\begin{aligned} S_{\mathrm{bos}} = \int \mathrm{d}^4x_A \mathrm{d}u\Big(&\tfrac{1}{4}\, A^{a-}_\mu \left(\mathcal{D}^{++}{}_a^b A_b^{-\mu} - 4\partial^\mu F_a^+\right) - M^{a-}\mathcal{D}^{++}{}_a^b N_b^- \\ &- P^{a(-3)}\left(D^{++}F_a^+ - \tfrac{1}{2}\,\partial_{a+}L^{+4}\right) \\ &+\mathrm{i}m\left(N^{a-}\partial_{a+}(F_b^+\lambda^{b+}) - M^{a-}\partial_{a+}(F_b^+\lambda^{b+})\right)\Big). \end{aligned} \tag{27}$$

The corresponding equations of motion read

$$\mathcal{D}^{++}M_a^- - \mathrm{i}m\,\partial_{a+}(F_b^+\lambda^{b+}) = 0\,, \tag{28}$$

$$\mathcal{D}^{++}N_a^- + \mathrm{i}m\,\partial_{a+}(F_b^+\lambda^{b+}) = 0\,, \tag{29}$$

along with (11) and (14). It is worth emphasizing that the equations of motion for M_a^- and N_a^- are modified as compared to (15) due to the mass deformation, while the equations (11) and (14) for F_a^+ and $A_{a\mu}^-$ are not modified. As in the massless case, these equations serve to express infinite sets of auxiliary fields collected in the involved quantities in terms of the central basis HK coordinate $f^{ai}(x)$ and its x–derivative.

After substituting these kinematical equations of motion into (27), the bosonic component action acquires the simple form

$$S_{\mathrm{bos}} = \int \mathrm{d}^4x_A \mathrm{d}u\left[-\frac{1}{2}\,A^{a-}_\mu\partial^\mu F_a^+ - \frac{\mathrm{i}m}{2}\,(M^{a-} - N^{a-})\partial_{a+}(F_b^+\lambda^{b+})\right]. \tag{30}$$

The harmonic fields in (30) are still solutions of (11), (14), (28) and (29). Since we have already solved the equations (11) and (14) while studying the massless case, the remaining equations to be solved are (28) and (29). The general form of the solution for M^{a-}, N^{a-} is given by

$$M^{a-} = -N^{a-} = \mathrm{i}m \left(\mathcal{F}_b^- \partial_+^a \lambda^{b+} + k^{ci} \left[\partial_{ci} \mathcal{F}^{a-} - 2\, E_{ci}^{a-} \right] \right) . \tag{31}$$

Here $\mathcal{F}_a^-$ is defined by

$$F_a^+ = \mathcal{D}^{++} \mathcal{F}_a^- = D^{++} \mathcal{F}_a^- - \tfrac{1}{2}\, \partial_{a+} \partial_{b+} L^{+4} \mathcal{F}^{b-} . \tag{32}$$

Substituting (19) and (31) into (30), we obtain,

$$S_{\mathrm{bos}} = \tfrac{1}{2} \int \mathrm{d}^4 x_A\, g_{ai,bj}(x_A)\, \partial^\mu f^{ai} \partial_\mu f^{bj} - \int \mathrm{d}^4 x_A V(f) \,, \tag{33}$$

$$V(f) = m^2 \int \mathrm{d}u\, \partial_{a+} (F_b^+ \lambda^{b+}) \Omega^{ad} \left[\mathcal{F}_e^- \partial_{d+} \lambda^{e+} + k^{ci} (\partial_{ci} \mathcal{F}_d^- - 2\Omega_{de} E_{ci}^{e-}) \right] . \tag{34}$$

The kinetic term in (33) has the same form as the massless bosonic action (22). Note that the potential in the generic case still displays a harmonic dependence while the kinetic term does not depend on the harmonic variables. The genuine scalar potential in x–space is obtained after performing the integration over harmonics. The u–integral can be performed under the sufficient condition, which is given in our original paper [30].

2.3 Massive HK sigma model in HSS: a multi−center case

In the previous subsection we derived the component action of the general massive HK nonlinear sigma model with at least one triholomorphic isometry. We did not specify the precise realization of this isometry. We obtained the kinetic term of the nonlinear sigma model which has exactly the form prescribed in Refs. [33, 11]. However, in the general case the harmonic integral in the scalar potential cannot be computed in a simple way. Fortunately, in the case of four–dimensional HK manifolds the situation is simplified radically due to the theorem [34, 28] claiming that any 4–dimensional HK metric with at least one $U(1)$ triholomorphic isometry falls into the class of Gibbons–Hawking multi–center metrics [35]. Moreover, it can be shown (see [26] and refs. therein) that the HK potentials for such metrics can always be brought to the form $L^{+4}_{mc} = L^{+4}(u^+ \cdot q^+, u)$ where $u^+ \cdot q^+ = u^{+a} q_a^+$ and the isometry is realized as the shift $q^{+a} \to q^{+a} + \varepsilon\, u^{+a}$. As a result, the computation of the potential is drastically simplified.

In the present case we have

$$\lambda^{a+} = u^{+a} \,, \quad \frac{\partial}{\partial x^5}\, q^{a+} = m\, u^{+a} . \tag{35}$$

In this particular case, the Lagrangian in (30) can be rewritten as follows

$$\mathcal{L} = \mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{pot}} \,, \tag{36}$$

$$\mathcal{L}_{\mathrm{kin}} = -\tfrac{1}{2} \int \mathrm{d}u A_\mu^{a-} \partial^\mu F_a^+ \,, \tag{37}$$

$$\mathcal{L}_{\mathrm{pot}} = -\mathrm{i}m \int \mathrm{d}u M^{a-} u_a^+ \,, \tag{38}$$

where we used $M_a^- = -N_a^-$, which follows from (28) and (29).

Our purpose is to derive the component action of physical bosons in x–space from the (x,u)–space action (36). One of the ways to obtain it is to substitute (35) into the general formula (33). However, it is easier to proceed directly by solving eqs. (11), (14), (28) and (29). We carry out this in two steps. First, we solve the equations of motion (11) and (14), and derive the kinetic term. As a result of solving these equations, F_a^+ and $A_a^{\mu-}$ are expressed in terms of the dynamical physical fields f^{ai}. It turns out that it is actually enough to solve equation for $A_a^{\mu-}$ partially, as distinct from the equation for F_a^+ which should be solved exactly. Secondly, we solve the equations (28) and (29). These solutions are needed to derive the scalar potential. We will see that the scalar potential is expressed in terms of the analytic HK potential after substituting the solutions into (38).

The equations of motion (11) and (14) are written in the considered particular case as

$$D^{++}F_a^+ - u_a^+ L^{+2} = 0\,, \tag{39}$$
$$D^{++}A_a^{\mu-} - u_a^+(u^+\cdot A^{\mu-}) - 2\partial^\mu F_a^+ = 0\,, \tag{40}$$

where

$$\begin{aligned} L^{+2}(u^+\cdot F^+, u) &= -\frac{1}{2}\,\frac{\partial L^{+4}}{\partial(u^+\cdot F^+)}\,, \\ L(u^+\cdot F^+, u) &= -\frac{1}{2}\,\frac{\partial^2 L^{+4}}{\partial(u^+\cdot F^+)^2}\,. \end{aligned} \tag{41}$$

First we solve (39). Substituting the following ansatz

$$F^{a+} = f^{ai}u_i^+ + v^{a+}(f^{ai}, u) \tag{42}$$

into (39), we obtain

$$\partial^{++}v^{a+}(f^{ai}, u) = u^{+a}L^{+2}\,. \tag{43}$$

Up to a gauge freedom, $v^{a+}(f)$ can be written as

$$v^{a+}(f^{ai}, u) = u^{+a}v(f^{ai}, u)\,. \tag{44}$$

Indeed,

$$u^+F^+ = u^{+a}(f_a{}^i u_i^+ + u_a^+ v(f^{ai}, u)) = u^{+a}f_a{}^i u_i^+\,. \tag{45}$$

Thus, eq. (43) amounts to

$$\partial^{++}v(f^{ai}, u) = L^{+2}(u^+\cdot f^+, u)\,. \tag{46}$$

Using the harmonic Green function [26], we obtain the general solution of this equation as

$$v(f^{ai}, u) = \int \mathrm{d}w\, \frac{u^+\cdot w^-}{u^+\cdot w^+}\, L^{+2}(w^+\cdot f^+, w)\,. \tag{47}$$

Thus, we find the final form of the solution of (39) to be

$$F^{a+} = f^{ai} u_i^+ + u^{+a} \int \mathrm{d}w \, \frac{u^+ \cdot w^-}{u^+ \cdot w^+} \, L^{+2}(w^+ \cdot f^+, w) \,, \tag{48}$$

which will be used for computing the kinetic term.

Next, we partially solve eq. (40). Multiplying (40) by u_a^+ and u_a^-, we obtain

$$D^{++}(u^+ \cdot \tilde{A}_\mu^-) = 0 \,, \tag{49}$$

$$D^{++}(u^- \cdot \tilde{A}_\mu^-) - (u^+ \cdot \tilde{A}_\mu^-)(1 - L) + 2u^{+a} \partial_\mu f_a{}^i u_i^- L + 2\partial_\mu v(f^{ai}, u) = 0 \,, \tag{50}$$

where

$$\tilde{A}_a^{\mu -} = A_a^{\mu -} - 2\partial^\mu f_a{}^i u_i^- \,. \tag{51}$$

Eq. (49) implies that $B^\mu(x) \equiv (u^+ \cdot \tilde{A}^{\mu -})$ does not depend on the harmonics. Substituting this into (50) and taking the harmonic integral of the l.h.s. of (50), we find

$$B^\mu(x) = -\frac{2}{1 + V_0} \, \partial^\mu f^{ai} V_{ai} \,, \tag{52}$$

where

$$V_{ai} = \int \mathrm{d}u \, u_a^+ u_i^- \, L(u^+ \cdot f^+, u) \,, \tag{53}$$

$$V_0 = \epsilon^{ai} V_{ai} = -\int du \, L(u^+ \cdot f^+, u) \,. \tag{54}$$

Here, we have used the property

$$\int \mathrm{d}u \, v(f, u) = \int \mathrm{d}u \, \mathrm{d}w \, \frac{u^+ \cdot w^-}{u^+ \cdot w^+} \, L^{+2}(w^+ \cdot f^+, w) = 0 \,, \tag{55}$$

which can be proved by representing u_i^+ in the numerator of the integrand as $u_i^+ = \partial_u^{++} u_i^-$, integrating by parts with respect to ∂_u^{++} and using the properties

$$(u^- \cdot w^-) \, \delta^{(1,-1)}(u, w) = 0$$

and

$$\int \mathrm{d}u \, \partial^{++} f^{(q)}(u) = 0 \,,$$

where q is a $U(1)$ charge. From (51) and (52), we obtain

$$A_{\mu a}^- = -\left[u_a^+ (u^- \cdot A_\mu^-) + 2\, u_a^- \left(\partial_\mu f^{bj} u_b^+ u_j^- + \frac{1}{1 + V_0} \, \partial_\mu f^{bj} \, V_{bj} \right) \right] . \tag{56}$$

Now we are ready to compute the kinetic term. As already mentioned, in order to compute the metric, there is no need to explicitly solve eq. (50) for the remaining unknown $(u^- \cdot A^-_\mu) = (u^- \cdot \tilde{A}^-_\mu) - 2\partial_\mu f^{ai} u^-_a u^-_i$. We substitute (48) and (56) into (37) and, integrating by parts with respect to D^{++}, obtain

$$\begin{aligned} \mathcal{L}_{\rm kin} = \int {\rm d}u \Big\{ & \frac{1}{2}\, D^{++}(u^- \cdot A^-_\mu)\partial^\mu f^{ai} u^+_{(a}u^-_{i)} \\ & - \left(\partial_\mu f^{ai} u^+_a u^-_i + \frac{1}{1+V_0}\,\partial_\mu f^{ai}\, V_{ai}\right)\left(\partial^\mu f^{bj} u^-_b u^+_j + \partial^\mu v\right)\Big\}. \end{aligned} \tag{57}$$

At this step, eq. (50) must be taken into account. We make use of it in the first term on the r.h.s. of (57), then perform the harmonic integral, integrate a few times by parts and use eqs. (55) and (46). Finally, we obtain the kinetic sigma model term just in the form (22) with

$$g_{ai,bj} = (1+V_0)\epsilon_{ab}\epsilon_{ij} + V_{ai}\epsilon_{bj} + V_{bj}\epsilon_{ai} + \frac{2}{1+V_0}\, V_{ai}V_{bj}\,. \tag{58}$$

The same metric has been earlier derived from the HSS approach in [31, 26]. There, the non-Lagrangian approach was used, with the inverse metric as the basic outcome:

$$g^{ai,bj} = \frac{1}{1+V_0}\left(\epsilon^{ab}\epsilon^{ij} + V^{ai}\epsilon^{bj} + V^{bj}\epsilon^{ai} + V^2\epsilon^{ai}\epsilon^{bj}\right), \tag{59}$$

where $V^2 = V^{ai}V_{ai}$. The Lagrangian approach used above is simpler and more direct. It can be easily employed to find the explicit form of the scalar potential term in (33) for the considered multi–center metrics.

To this end, we should still solve eqs. (28) and (29) which in this particular case have the following form

$$\mathcal{D}^{++}M^-_a + {\rm i}m\, u^+_a = 0\,, \tag{60}$$

$$\mathcal{D}^{++}N^-_a - {\rm i}m\, u^+_a = 0\,. \tag{61}$$

Thus eqs. (60) and (61) are reduced to the single equation

$$D^{++}M^-_a - u^+_a\,(u^+ \cdot M^-)\,L + {\rm i}m\, u^+_a = 0\,. \tag{62}$$

Introducing

$$\tilde{M}^{a-} = M^{a-} + {\rm i}m\, u^{-a} \tag{63}$$

and projecting (62) on the harmonics u^+_a and u^-_a, respectively, we obtain

$$D^{++}(u^+ \cdot \tilde{M}^-) = 0\,, \tag{64}$$

$$D^{++}(u^- \cdot \tilde{M}^-) - (u^+ \cdot \tilde{M}^-)(1-L) - {\rm i}mL = 0\,. \tag{65}$$

It follows from (64) that $(u^+ \cdot \tilde{M}^-)$ does not depend on harmonics,

$$(u^+ \cdot \tilde{M}^-) = A(x)\,. \tag{66}$$

Substituting this into (65) and integrating the l.h.s. of the latter over harmonics, we obtain

$$A(x) = \mathrm{i}m\,\frac{V_0}{1+V_0} \;\Rightarrow\; (u^+ M^-) = A - \mathrm{i}m = -\mathrm{i}m\,\frac{1}{1+V_0}\,. \tag{67}$$

Substituting (67) into (38), we find the final form of the scalar potential to be

$$\mathcal{L}_{\rm pot}(x,u) = -m^2\,\frac{1}{1+V_0}\,, \quad \partial^{++}\mathcal{L}_{\rm pot} = 0\,. \tag{68}$$

Thus, we have managed to solve the equations of motion (11), (14), (28) and (29) and so have found the explicit form of the component bosonic action (33) for the multi–center case. Both the HK metric and potential term in (33) are expressed, by eqs. (58) and (68), in terms of the single object, multi–center potential V_0 defined in (54).

Let us derive the same result (68) in another way. By using (7) and (21), we obtain

$$\delta F^{a+} = \varepsilon \partial_{bk} F^{a+} k^{bk} = -\varepsilon E^{a+}_{bk} k^{bk}\,, \tag{69}$$

where k^{ck} is usual Killing vector defined by $\delta f^{ck} = \varepsilon k^{ck}$. Applying the formula (69) to the particular case (35) and taking into account that it follows from eq. (48) that $\delta f^{ai} = \varepsilon\,\epsilon^{ai} \;\Rightarrow\; k^{ai} = \epsilon^{ai}$, we find

$$u^{+a} = -E^{a+}_{bi} k^{bi} = -E^{a+}_{bi}\,\epsilon^{bi} = \mathcal{D}^{++} E^{a-}_{bi}\,\epsilon^{bi}\,. \tag{70}$$

Then we can rewrite the Lagrangian (38) as

$$\mathcal{L}_{\rm pot} = -\mathrm{i}m\, M^{a-}\Omega_{ac} E^{c+}_{bi}\epsilon^{bi}\,. \tag{71}$$

Moreover, it is easy to find

$$M^{a-} = -\mathrm{i}m\, E^{a-}_{bi}\epsilon^{bi}\,. \tag{72}$$

Now we substitute this into (71), use the definition (23) and take into account eq. (21) and the fact that under the u–integral one can integrate by parts. As the result we obtain

$$\mathcal{L}_{\rm pot} = -\frac{m^2}{2}\, g_{ai,bk}\epsilon^{ai}\epsilon^{bk} = -\frac{m^2}{2}\, g_{ai,bk}\, k^{ai}k^{bk}\,, \tag{73}$$

which is just the square of the norm of Killing vector $k^{ai} = \epsilon^{ai}$. Using (58), we find

$$g_{ai,bk}\epsilon^{ai}\epsilon^{bk} = \frac{2}{1+V_0}\,, \tag{74}$$

i.e., we come back to the expression (68). This means that the scalar potential is determined by the norm of the Killing vector. This fact was originally obtained in [14] by means of an on-shell formalism.

Finally, let us show that eqs. (58) and (68) can be put in the standard multi–center form. Introducing,

$$\vec{V} = -\frac{\mathrm{i}}{2}\,(\vec{\tau})^{ai}V_{ai}\,, \quad \vec{X} = \frac{\mathrm{i}}{\sqrt{2}}\,(\vec{\tau})^{ai}f_{ai}\,, \quad \varphi = \frac{1}{\sqrt{2}}\,\epsilon_{ai}f^{ai}\,, \quad U = 1 + V_0\,, \tag{75}$$

where $\vec{\tau}^{ai}$ are the Pauli matrices,[4] we can write down Lagrangian (36) in the form

$$\mathcal{L} = \tfrac{1}{2}\left\{U\partial_\mu\vec{X}\cdot\partial^\mu\vec{X} + U^{-1}\mathcal{D}_\mu\varphi\mathcal{D}^\mu\varphi - m^2U^{-1}\right\}, \tag{76}$$

where m has been changed by $m/\sqrt{2}$, and $\vec{X} = (X^1, X^2, X^3)$, φ are real scalar fields, and $\mathcal{D}_\mu\varphi = \partial_\mu\varphi + \vec{V}\cdot\partial_\mu\vec{X}$. The fields $\vec{V}$ and $1 + V_0$, by their definition, satisfy the differential equations

$$\vec{\nabla}\times\vec{V} = \vec{\nabla}U\,, \quad \Delta U = 0\,, \quad \frac{\partial}{\partial\varphi}(1 + V_0) = 0\,. \tag{77}$$

The scalar potential is given by

$$V = m^2U^{-1}\,. \tag{78}$$

This precisely coincides with what has been found in [14]. In this parameterization, the $U(1)$ isometry (35) is realized as a shift of the coordinate φ of the HK manifold:

$$\delta\varphi = \sqrt{2}\,\varepsilon\,, \quad \delta\vec{X} = 0\,. \tag{79}$$

Note that it is possible to extend the above consideration to the case of $4n$ dimensional HK manifolds whose metric have n commuting translation isometries, i.e., to the general case of toric HK metrics. In this case, the ansatz for the analytic HK potential is $L^{+4}_{\mathrm{toric}} = L^{+4}(u^{+i}q^+_{ai}, u)$ where $a = 1, \cdots, n$ [26].

Now we consider two particular examples.

i) *Taub–NUT case*

The analytic potential in the Taub–NUT (TN) case can be chosen as [5]

$$L^{+4}_{\mathrm{TN}}(u^+\cdot q^+, u) = \frac{2}{\lambda}\,(g^{++})^2 = \frac{2}{\lambda}\left(\frac{L^{++} - c^{++}}{1 + \sqrt{1 + (L^{++} - c^{++})c^{--}}}\right)^2, \qquad L^{++} = u^+\cdot q^+\,, \tag{80}$$

where $c^{\pm\pm} = c^{ij}u^\pm_i u^\pm_j$, $c^2 = \frac{1}{2}c^{ij}c_{ij} = 1$ and λ is a constant. The corresponding potential V_0 is given by

$$V_0 = \frac{1}{2}\int \mathrm{d}u\,\frac{\partial^2 L^{+4}_{\mathrm{TN}}}{\partial(L^{++})^2} = \frac{1}{2\lambda}\int \mathrm{d}u\,\frac{1}{\left[1 + (L^{++} - c^{++})c^{--}\right]^{3/2}}\,. \tag{81}$$

[4]) Here $\vec{\tau}^{ai} = \epsilon^{ab}\vec{\tau}_b{}^i$ where $\vec{\tau}_a{}^i$ are the standard Pauli matrices.

[5]) The form of the TN and Eguchi–Hanson (EH) HK potentials can be found in Chapter 6.6.1 of Ref. [26] (see eqs. (6.72) and (6.73) there). The HK potential for the double Taub–NUT metric is obtained by a slight modification of these potentials.

To compute the harmonic integral in (81), we make use of the general formula derived in Ref. [36]:

$$\int \mathrm{d}u \, \frac{1}{[1+(G^{++}-c^{++})c^{--}/c^2]^{3/2}} = \frac{\sqrt{c^{ik}c_{ik}}}{\sqrt{G^{ik}G_{ik}}}\,, \tag{82}$$

where

$$G^{++} = G^{ik}u_i^+u_k^+\,. \tag{83}$$

Using this general formula and choosing the particular $SU(2)_R$ frame, $c^{11}=c^{22}=0$, $c^{12}=\mathrm{i}$, it is easy to find

$$U_{\mathrm{TN}} = 1 + V_0 = 1 + \frac{1}{\sqrt{2}\,\lambda}\,\frac{1}{\sqrt{L^{ik}L_{ik}}}\,. \tag{84}$$

We can rewrite (84) in terms of the multi–center coordinate $\vec{X}$. Using the relations

$$\begin{aligned} L^{++} = u^{+a}F_a^+ = -f^{ai}u_a^+u_i^+ = -f^{(ai)}u^+_{(a}u^+_{i)}\,, \\ \vec{X}\cdot\vec{X} = f_{(ai)}f^{(ai)}\,, \end{aligned} \tag{85}$$

we find

$$U_{\mathrm{TN}} = 1 + \frac{1}{\sqrt{2}\,\lambda}\,\frac{1}{|\vec{X}|}\,. \tag{86}$$

This form of the one–center TN potential corresponds to the center located at $\vec{X}=0$. In the following example, for later convenience, we choose another position of the center.

ii) *Double Taub–NUT case*

Now we consider more general double Taub–NUT (DTN) case. The relevant analytic HK potential reads

$$\begin{aligned} L^{+4}_{\mathrm{DTN}}(u^+\cdot q^+, u) = 2\left(\frac{L^{++}}{1+\sqrt{1+L^{++}\eta^{--}}}\right)^2 + \frac{2}{\gamma}\left(\frac{L^{++}}{1+\sqrt{1-L^{++}\eta^{--}}}\right)^2 \\ -(1-a)(L^{++})^2\,, \end{aligned} \tag{87}$$

where a and γ are some constants and $\eta^{\pm\pm}=\eta^{ik}u_i^\pm u_i^\pm$. If $\eta^{ik}=0$ we return to the TN case. For $\eta^{ik}\neq 0$, one can always choose $\eta^2=\frac{1}{2}\eta^{ij}\eta_{ij}=1$ by the appropriate rescaling of q^{a+}. As we will see, η^{ij} specifies the location of the centers. The potential V_0 in the present case reads

$$V_0 = \frac{1}{2}\int \mathrm{d}u\left(\frac{1}{[1+L^{++}\eta^{--}]^{3/2}} + \frac{1}{\gamma}\,\frac{1}{[1-L^{++}\eta^{--}]^{3/2}}\right) - (1-a)\,, \tag{88}$$

whence

$$\begin{aligned} U_{\mathrm{DTN}} = 1 + V_0 = a + \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{(L^{ik}+\eta^{ik})(L_{ik}+\eta_{ik})}} \\ + \frac{1}{\sqrt{2}\,\gamma}\,\frac{1}{\sqrt{(L^{ik}-\eta^{ik})(L_{ik}-\eta_{ik})}}\,. \end{aligned} \tag{89}$$

Introducing,

$$\vec{\xi} = \frac{\mathrm{i}}{\sqrt{2}}\,\vec{\tau}^{ij}\eta_{ij}\,, \tag{90}$$

and using (85), we can rewrite (89) as

$$U_{\mathrm{DTN}} = a + \frac{1}{\sqrt{2}}\,\frac{1}{|\vec{X}+\vec{\xi}|} + \frac{1}{\sqrt{2}\,\gamma}\,\frac{1}{|\vec{X}-\vec{\xi}|}\,. \tag{91}$$

For $a \neq 0$ this is a two–center ALE potential, with the constant vector $\vec{\xi}$ specifying the position of both centers (they are collinear to each other). For $a = 0$, the potential U_{DTN} becomes the general EH potential with non-equal "masses". Like the DTN potential itself, its EH limiting case possesses only $U(1)\times U(1)$ isometry in contrast to the $SU(2)\times U(1)$ isometry of the standard EH potential which is recovered under the choice $\gamma = 1$.

3 Structure of SUSY vacua and the BPS equation

From the form of the scalar potential (68) we can find the SUSY vacuum condition which is similar to that in the $\mathcal{N}=1$ case. The condition of SUSY vacuum is the vanishing of the scalar potential

$$0 = V(f^{ai}) = \frac{m^2}{1+V_0} = g_{ai,bj}k^{ai}k^{bj}\,. \tag{92}$$

We find that in our parameterization of the multi–center case the SUSY vacuum exists, provided there is a point where the potential V_0 goes to infinity,

$$V_0 = \tfrac{1}{2}\int \mathrm{d}u\,\frac{\partial^2 L^{+4}}{\partial (L^{++})^2} \to \infty\,. \tag{93}$$

We expect that this condition imposes strong restrictions on the original HK potential L^{+4}, though for the time being we do not know them in full generality. Now we apply eq. (93) to the previous examples.

i) *TN case*

In the TN case, it follows from (86) that the condition (93) can be realized only for the vacuum expectation value $\vec{X} = 0$. Thus the theory has only one SUSY vacuum. As a consequence, no domain wall solution can be found in this case since the existence of the domain wall solutions requires that the theory has at least two vacua.

ii) *DTN case*

In the DTN case, the theory has two discrete vacua which are realized at vacuum expectation values $\vec{X} = -\vec{\xi}$ and $\vec{X} = \vec{\xi}$. Indeed, in this case there exists the domain wall solution as we shall see soon.

In order to find the behaviour of the potential in the DTN case, we take $\vec{\xi} = (0,0,\xi)$ and introduce the spherical coordinates such as

$$X^1 = r\sin\Theta\cos\Psi\,,\;\; X^2 = r\sin\Theta\sin\Psi\,,\;\; X^3 = \sqrt{r^2+\xi^2}\cos\Theta\,,\;\; \varphi = \Phi+\Psi\,. \tag{94}$$

In this parameterization, the geometrical meaning of the target manifold becomes clear: the fields r and Θ are the coordinates of base manifold S^2 and Φ and Ψ form a fiber over this base manifold. The DTN potential (91) takes the following form in the coordinates (94):

$$U_{\rm DTN} = a + \frac{1}{\sqrt{2}}\left(\frac{1}{\left|\sqrt{r^2+\xi^2}+\xi\cos\Theta\right|} + \frac{1}{\gamma}\,\frac{1}{\left|\sqrt{r^2+\xi^2}-\xi\cos\Theta\right|}\right). \tag{95}$$

Note that the potential (95) depends only on the real coordinates r and Θ, and not on Φ and Ψ, which reflects the presence of $U(1)\times U(1)$ isometry. So the vacuum configurations "live" on the submanifold S^2 of the full target space.

The vacuum expectation values, for instance, $\vec{X}=0$ in the TN case, can be easily cast in the HSS language using (85); these amount to $L^{++}=0$ in the TN case and to $L^{++}=-\eta^{++}$ and $L^{++}=\eta^{++}$ in the DTN case. For these special values of L^{++}, the analytic potentials (80) and (87), equally as their second derivatives entering (81) and (88), acquire singularities at some points of the harmonic sphere $S^2 \sim \{u^+_i, u^-_k\}$. As the result, the u–integral (93) which specifies the sufficient condition for vacuum to be the SUSY one becomes divergent. E.g., in the TN case, substituting $L^{++}=0$ ($L^{ij}=0$) into (93), one obtains harmonic integral

$$V_0\Big|_{L^{ij}=0} = \tfrac{1}{2}\int {\rm d}u\,\frac{1}{[1-c^{++}c^{--}]^{3/2}} \tag{96}$$

which is divergent. The same divergent integrals are obtained in the DTN case for two values of L^{++}, namely for $L^{++}=-\eta^{++}$ and $L^{++}=\eta^{++}$. The existence of two discrete vacua in the latter case guarantees the existence of the domain wall solution.

In the following, we consider the general BPS equation and apply it to the DTN case. If we assume that there is a non-trivial configuration along the spatial y direction and this configuration is static, the energy density can be written as

$$\begin{aligned}\mathcal{E} &= \tfrac{1}{2}\,U\partial_2\vec{X}\cdot\partial_2\vec{X} + \frac{1}{2}\,U^{-1}\mathcal{D}_2\varphi\mathcal{D}_2\varphi + \tfrac{1}{2}\,m^2U^{-1}\\ &= \tfrac{1}{2}\,U(\partial_2\vec{X}-mU^{-1}\vec{n})\cdot(\partial_2\vec{X}-mU^{-1}\vec{n}) + \partial_2\vec{X}\cdot\vec{n} + \tfrac{1}{2}\,U^{-1}\mathcal{D}_2\varphi\mathcal{D}_2\varphi\\ &\geq \partial_2\vec{X}\cdot\vec{n}\,,\end{aligned} \tag{97}$$

where $\vec{n}$ is a unit vector. BPS equation is easily read off as

$$\partial_2\vec{X} - mU^{-1}\vec{n} = 0\,, \tag{98}$$

$$\mathcal{D}_2\varphi = 0\,. \tag{99}$$

Taking $\vec{n}=(0,0,1)$ and using $\vec{n}\cdot\vec{V}=0$ [37], BPS equation is simplified to

$$\frac{\partial\varphi}{\partial y} = 0\,,\quad \frac{\partial X^1}{\partial y} = \frac{\partial X^2}{\partial y} = 0\,, \tag{100}$$

$$\frac{\partial X^3}{\partial y} = mU^{-1}\,. \tag{101}$$

Eq. (100) can be easily solved as $\varphi = \text{const}$, $X^1 = \text{const}$, $X^2 = \text{const}$. Without loss of generality, these constants can be put equal to zero, i.e., $\varphi = X^1 = X^2 = 0$. Using these solutions and substituting (91) into (101), we bring eq. (101) to the form

$$\frac{\partial X}{\partial y} = m \, \frac{\sqrt{2}\gamma(\xi^2 - X^2)}{\sqrt{2}\, a\gamma(\xi^2 - X^2) + \gamma(\xi - X) + \xi + X}\,, \qquad X \equiv X^3\,. \tag{102}$$

This equation can be easily solved. Figs. 1 and 2 show the profiles of the domain wall solutions. Fig. 1 shows the profiles for some values of γ with $a = 1$. For $\gamma = 1$, the metric becomes the DTN metric with equal masses and the scalar configuration is symmetric at the center of the wall for $y = 0$. In particular, for $\gamma = 1$ and $a = 0$ (the standard EH case), analytic solution is obtained as

$$X = \xi \tanh\left(\frac{\sqrt{2}\, m}{2\xi}\,(y + y_0)\right), \tag{103}$$

where y_0 is an integration constant which specifies the position of the wall. As γ deviates from the value $\gamma = 1$, the scalar configuration becomes asymmetric. For fixed γ, the behaviour of the solution is shown in Fig. 2. As was mentioned, when $\gamma \to \infty$, the metric approaches the TN metric and therefore the domain wall solution does not exist in this limit.

4 Summary and concluding remarks

We have studied $\mathcal{N} = 2$ massive nonlinear sigma model starting from the action in the off-shell HSS formulation which manifests the full $\mathcal{N} = 2$ SUSY. The scalar potential was obtained by assigning to q^+ a dependence on the central charge coordinate x_5, such that $\partial/\partial x_5$ is identified with the Killing vector of the isometry. The component bosonic action was obtained based on the one-to-one correspondence between the Lagrangian and non-Lagrangian approaches to the HK geometry. As was shown in [26], the kinetic term at the component level in the general nonlinear sigma model is composed of the vielbeins and has a form which is independent of the harmonic variables. On the other hand, the scalar potential in the general case of one isometry still involves an integration over harmonics. Its more preferable form, which does not contain the harmonic integral, was derived in our paper [30].

Massive nonlinear sigma models with multi–center metrics were examined. In the generic HK case, solving the kinematical part of the equations of motion is very difficult problem. However, in the multi–center case, the situation is much simpler. We solved the kinematical part of the equations of motion and obtained the physical component action where integration over harmonic variables was performed to the end. It was shown that both the target metric and the scalar potential can be expressed in terms of the single analytic HK potential. The scalar potential was found to be fully specified by the norm of the Killing vector, which is in agreement with the earlier derivation of Ref. [14]. Given the explicit form of the scalar potential, we discussed the SUSY vacuum condition. The SUSY vacuum condition

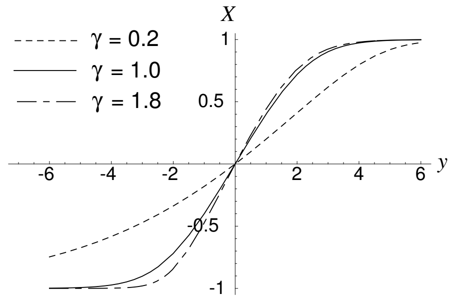

Fig. 1. Wall solutions for some values of γ with $a = 1$.

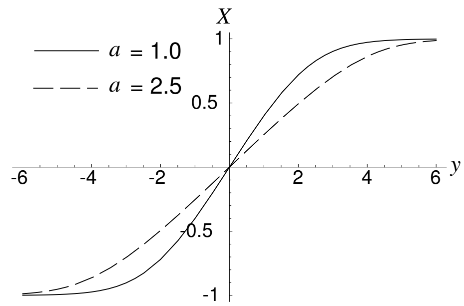

Fig. 2. Wall solutions for some values of a with $\gamma = 1$.

was related to the analytic HK potential. This result is the $\mathcal{N} = 2$ extension of the similar condition in $\mathcal{N} = 1$ theory which involves the Kähler potential and the superpotential. We derived BPS equation in the general multi–center case and BPS domain wall solution was obtained for the DTN case.

References

[1] P. Hořava and E. Witten: Nucl. Phys. B **460** (1996) 506, [`hep-th/9510209`].

[2] N. Arkani-Hamed, S. Dimopoulos and G. Dvali: Phys. Lett. B **429** (1998) 263, [`hep-ph/9803315`];
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali: Phys. Lett. B **436** (1998) 257, [`hep-ph/9804398`].

[3] L. Randall and R. Sundrum: Phys. Rev. Lett. **83** (1999) 3370, [`hep-ph/9905221`]; Phys. Rev. Lett. **83** (1999) 4690, [`hep-th/9906064`].

[4] V.A. Rubakov and M.E. Shaposhnikov: Phys. Lett. B **125** (1983) 136.

[5] O. Klein: Z. Phys. **37** (1926) 895.

[6] E. Bogomol'nyi: Sov. J. Nucl. Phys. B **24** (1976) 449;
M.K. Prasad and C.H. Sommerfield: Phys. Rev. Lett. **35** (1975) 760.

[7] E. Witten and D. Olive: Phys. Lett. B **78** (1978) 97.

[8] M. Arai, S. Fujita, M. Naganuma and N. Sakai: Phys. Lett. B **556** (2003) 192, [hep-th/0212175]; An exact BPS wall solution in Five Dimensional Supergravity, to be published in the *Proceedings of the International Workshop on "Supersymmetries and Quantum Symmetries" (SQS'03)*, [hep-th/0311210].

[9] M. Eto, S. Fujita, M. Naganuma and N. Sakai: Phys. Rev. D **68** (2004) 025007, [hep-th/0306198].

[10] Y. Isozumi, M .Nitta, K. Ohashi and N. Sakai: *Construction of Non-Abelian Walls and their Complete Moduli Space*, [hep-th/0404198]; *All Exact Solutions of a 1/4 Bogomol'nyi–Prasad–Sommerfield Equation*, [hep-th/0405129]; *Non-Abelian Walls in Supersymmetric Gauge Theories*, [hep-th/0405149].

[11] L. Alvarez-Gaumé and D.Z. Freedman: Comm. Math. Phys. **80** (1981) 443.

[12] J. Scherk and J.H. Schwarz: Phys. Lett. B **82** (1979) 60; Nucl. Phys. B **153** (1979) 61.

[13] L. Alvarez-Gaumé and D.Z. Freedman: Comm. Math. Phys. **91** (1983) 97.

[14] J.P. Gauntlett, D. Tong and P.K. Townsend: Phys. Rev. D **63** (2001) 085001, [hep-th/0007124].

[15] E. Abraham and P.K. Townsend: Phys. Lett. B **291** (1992) 85.

[16] J.P. Gauntlett, R. Portugues, D. Tong and P.K. Townsend: Phys. Rev. D **63** (2001) 085002, [hep-th/0008221].

[17] J.P. Gauntlett, D. Tong and P.K. Townsend: Phys. Rev. D **64** 2001) 025010, [hep-th/0012178].

[18] M. Naganuma, M. Nitta and N. Sakai: Grav. Cosmol. **8** (2002) 129, [hep-th/0108133].

[19] R. Portugues and P.K. Townsend: JHEP **0204** (2002) 039, [hep-th/0203181].

[20] D. Tong: Phys. Rev. D **66** (2002) 025013, [hep-th/0202012].

[21] M. Arai, M. Naganuma, M. Nitta and N. Sakai: Nucl. Phys. B **652** (2003) 35, [hep-th/0211103]; BPS Wall in $N = 2$ SUSY Nonlinear Sigma Model with Eguchi–Hanson Manifold in *Garden of Quanta — In honor of Hiroshi Ezawa*, Eds. by J. Arafune et al., World Scientific Publishing Co. Pte. Ltd. Singapore, 2003, pp299 – 325, [hep-th/0302028].

[22] M. Arai, M. Nitta and N. Sakai: *Vacua of Massive Hyper-Kähler Sigma Models of Non-Abelian Quotient*, [hep-th/0307274]; BPS Domain Walls in Massive Hyper-Kahler Sigma Models, to be published in the *Proceedings of the 3rd International Symposium on Quantum Theory and Symmetries (QTS3)*, [hep-th/0401084]; Massive Hyper–Kähler Sigma Models and BPS Domain Walls, to be published in the *Proceedings of the International Conference on Symmetry Methods in Physics (SYMPHYS10)*, [hep-th/0401102].

[23] M. Naganuma, M. Nitta and N. Sakai: Phys. Rev. D **65** (2002) 045016, [hep-th/0108179].

[24] Y. Sakamura: Nucl. Phys. B **656** (2003) 132, [hep-th/0207159]; JHEP **0304** (2003) 008, [hep-th/0302196].

[25] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev: Class. Quantum Grav. **1** (1984) 469;
A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev: Class. Quantum Grav. **2** (1985) 601; Class. Quantum Grav. **2** (1985) 617.

[26] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev: *Harmonic Superspace.* Cambridge University Press, Cambridge, 2001, 306p.

[27] A. Taub: Ann. Math. **53** (1951) 472;
E. Newman, L. Tamburino and T. Unti: J. Math. Phys. **4** (1963) 915;
S.W. Hawking: Phys. Lett. A **60** (1977) 81.

[28] G.W. Gibbons, D. Olivier, P.I. Ruback and G. Valent: Nucl. Phys. B **296** (1988) 679.

[29] T. Eguchi and A.J. Hanson: Phys. Lett. B **74** (1978) 249; Ann. Phys. **120** (1979) 82.

[30] M. Arai, J. Niederle and E. Ivanov: Nucl. Phys. B **680** (2004) 23, [hep-th/0312037].

[31] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev: Ann.Phys. **185** (1988) 1; Ann. Phys. **185** (1988) 22.

[32] N. Ohta, H. Sugata and H. Yamaguchi: Ann. Phys. **172** (1986) 26.

[33] B. Zumino: Phys. Lett. B **87** (1973) 203.

[34] G.W. Gibbons and P.J. Ruback: Comm. Math. Phys. **115** (1988) 267.

[35] G.W. Gibbons and S.W. Hawking: Phys. Lett. B **78** (1978) 430.

[36] A. Galperin, E. Ivanov and V. Ogievetsky: Phys. Scr. **T15** (1987) 176.

[37] G. Papadopoulos and P.K. Townsend: Nucl. Phys. B **444** (1995) 245, [hep-th/9501069].