On $sl(N)$ and $sl(M|N)$ integrable open spin chains

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We study open spin chains based on rational $sl(N)$ and $sl(M|N)$ R–matrices. We classify the solutions of the reflection equations, for both the soliton–preserving and soliton–non–preserving cases. We then write the Bethe equations for these open spin chains.

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1 Introduction

We are interested in open quantum spin chains based on rational R–matrices of $sl(N)$ and $sl(M|N)$:

$$R_{12}(\lambda) = \lambda \mathbb{I} + \mathrm{i} P_{12}\,, \tag{1}$$

where P is the super–permutation operator

$$P = \sum_{i,j=1}^{M+N} (-1)^{[j]} E_{ij} \otimes E_{ji}\,. \tag{2}$$

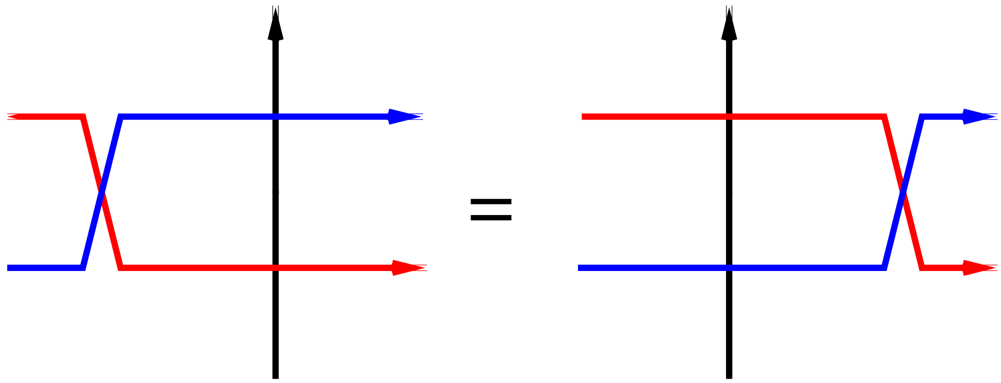
We will give a classification of the reflection matrices compatible with the integrability of the open spin chain, in the two cases of soliton preserving and soliton non-preserving boundary conditions.

In section 2, we recall graphically the proof of commutation of transfer matrices for closed chains. We then recall in section 3 the commutation for open chains. In section 4, we define the transfer matrix for open spin chains with soliton non preserving boundary conditions. In section 5 we give the classification of solutions of the reflection equations, for both soliton preserving and soliton non preserving cases. We finally present the analytical Bethe ansatz method for these chains and end with the Bethe equations. More details and references can be found in [1].

2 Closed chain integrability

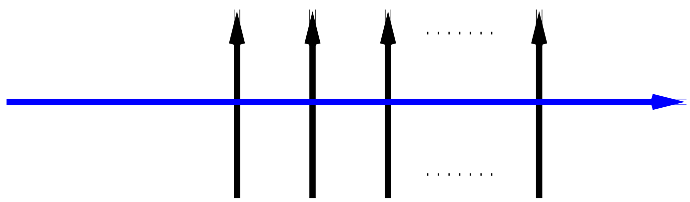
Let R be a solution of the Yang–Baxter equation

$$R_{12}(\lambda_1-\lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1-\lambda_2)\,. \tag{3}$$



On a chain with L sites, we define the monodromy matrix as

$$T(\lambda) = R_{a1}(\lambda)\ R_{a2}(\lambda)\ \cdots\ R_{aL}(\lambda) \tag{4}$$



and the transfer matrix as its *super trace*

$$t(\lambda) = \mathrm{Tr}_a\ T(\lambda)\,. \tag{5}$$

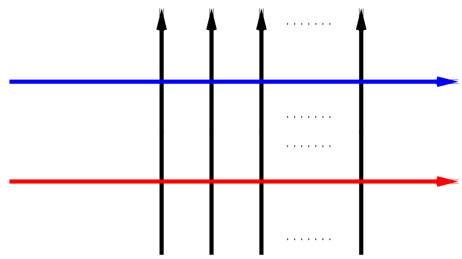
The Hamiltonian is one of the terms of the expansion of the transfer matrix

$$\mathcal{H} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}t(\lambda)\Big|_{\lambda=0}\,. \tag{6}$$

The main property used for integrability of closed spin chains, i.e. commutation of the transfer matrix for different values of the spectral parameter, is the *local* Yang–Baxter equation.

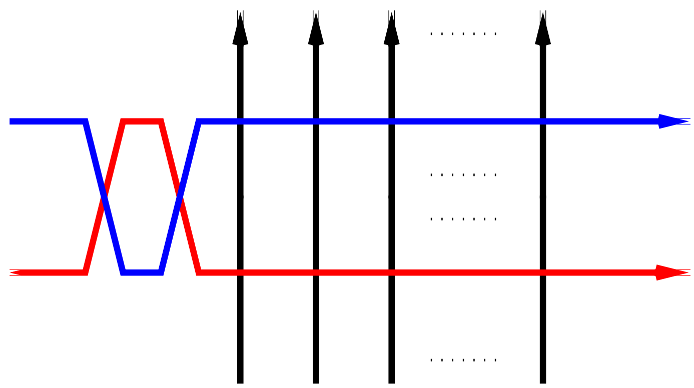
Graphical proof of the commutation of transfer matrices $t(u)$ and $t(v)$:

$$t(u)\,t(v) = Tr\ \ Tr$$

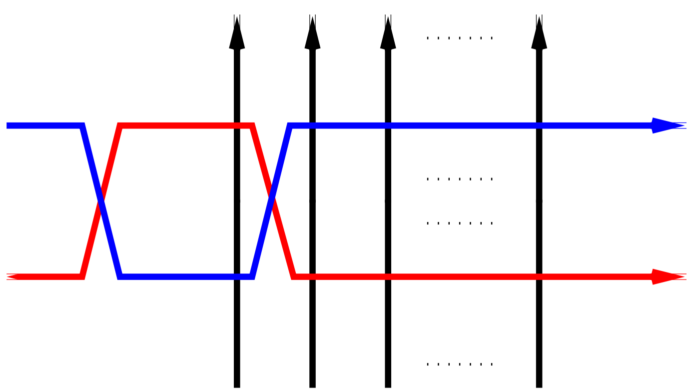


Insertion of $R_{ab}(u-v)R_{ab}^{-1}(u-v) \longrightarrow$

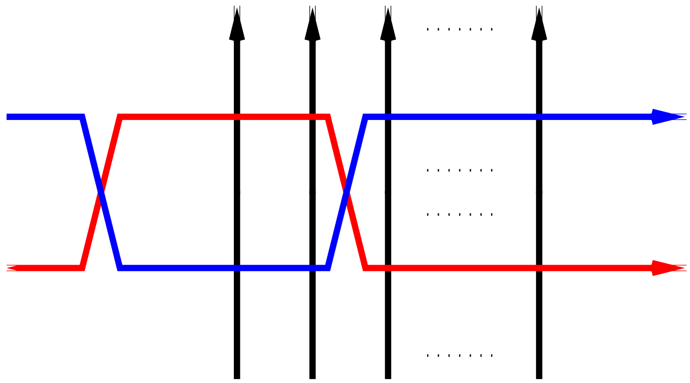
$=$



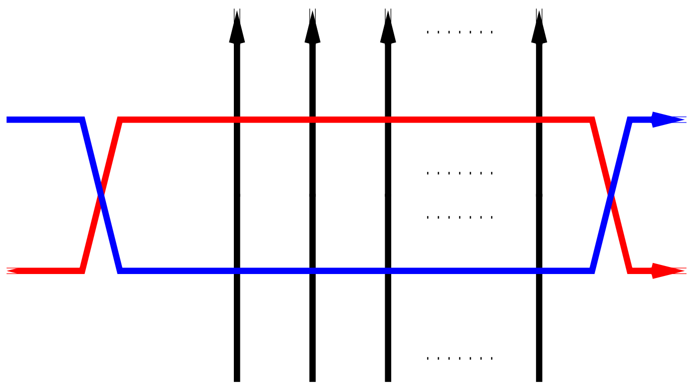
Use of Yang–Baxter $\longrightarrow$



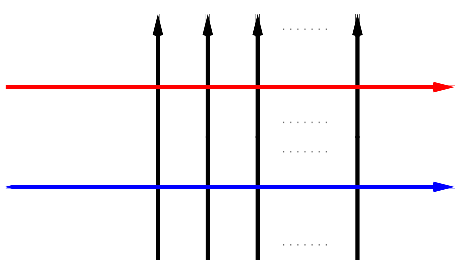
Use of Yang–Baxter again $\longrightarrow$



again ...



Cyclicity of trace $\longrightarrow$

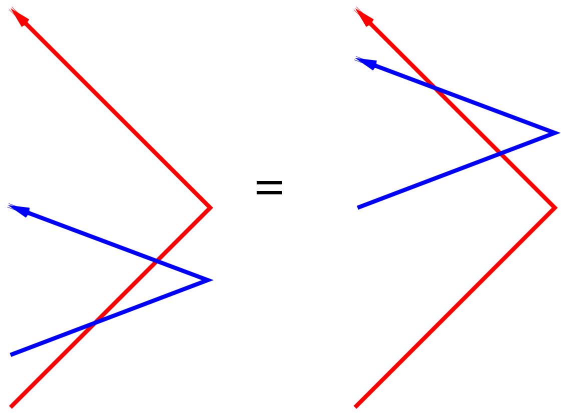


$= t(v)\, t(u)$

3 Open chain integrability

For the integrability of open chains, one also needs the (local) reflection equation:

$$\begin{aligned} &R_{ab}(\lambda_a-\lambda_b)K_a(\lambda_a)R_{ba}(\lambda_a+\lambda_b)K_b(\lambda_b)= \\ &\qquad = K_b(\lambda_b)R_{ab}(\lambda_a+\lambda_b)K_a(\lambda_a)R_{ba}(\lambda_a-\lambda_b)\,. \end{aligned} \tag{7}$$



Let

$$T_a(\lambda)=R_{aL}(\lambda)R_{a,L-1}(\lambda)\cdots R_{a2}(\lambda)R_{a1}(\lambda) \tag{8}$$

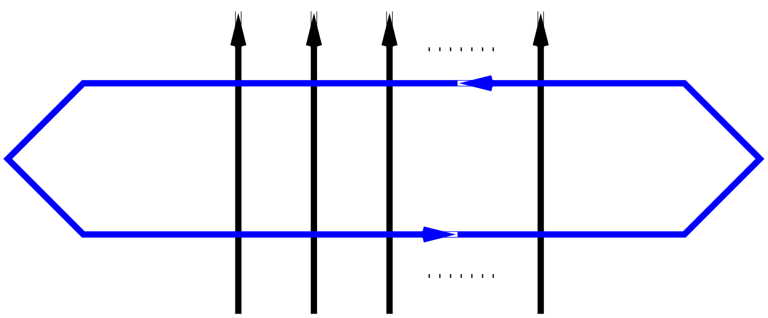
and

$$\hat{T}_a(\lambda)=R_{1a}(\lambda)R_{2a}(\lambda)\cdots R_{L-1,a}(\lambda)R_{La}(\lambda)\,. \tag{9}$$

The open spin chain transfer matrix is now defined as the super trace:

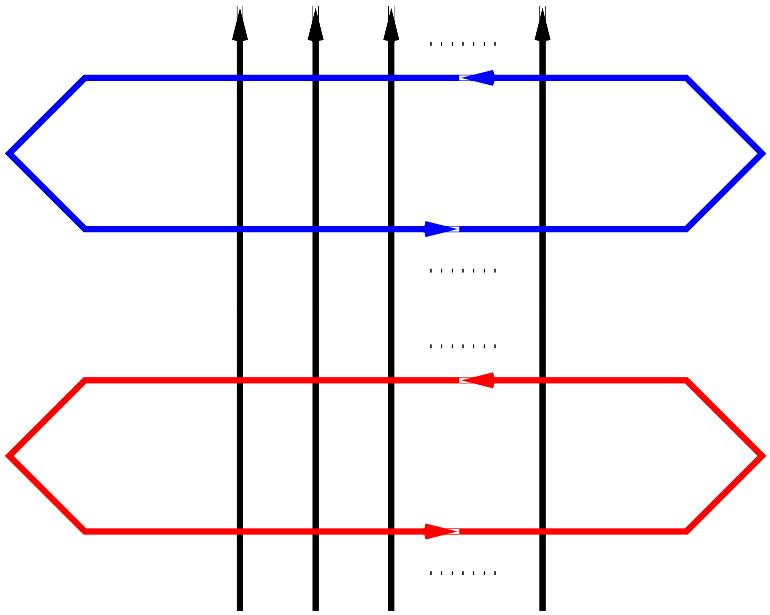
$$t(\lambda)=\mathrm{Tr}_a\,K_a^+(\lambda)\;T_a(\lambda)\;K_a^-(\lambda)\;\hat{T}_a(\lambda) \tag{10}$$

$t(\lambda)=$

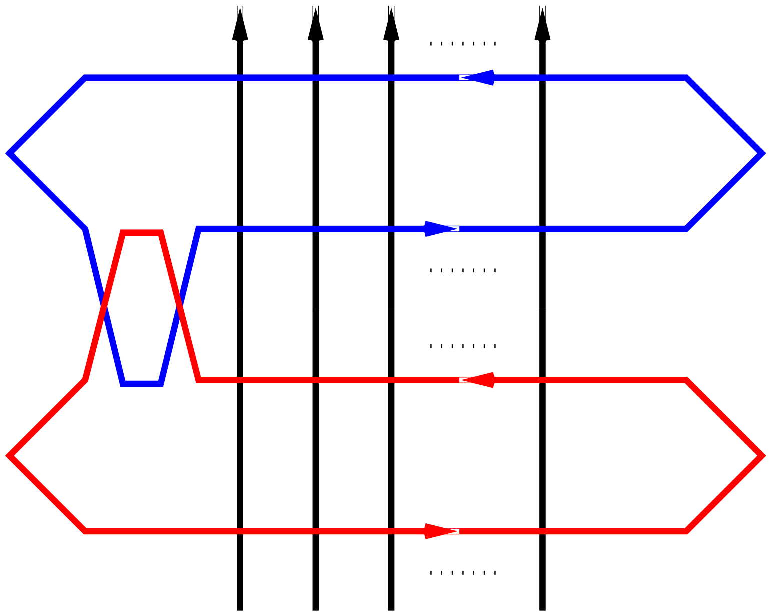


Graphical proof of the commutation of transfer matrices for different spectral parameters, following Cherednik [2] and Sklyanin [3]:

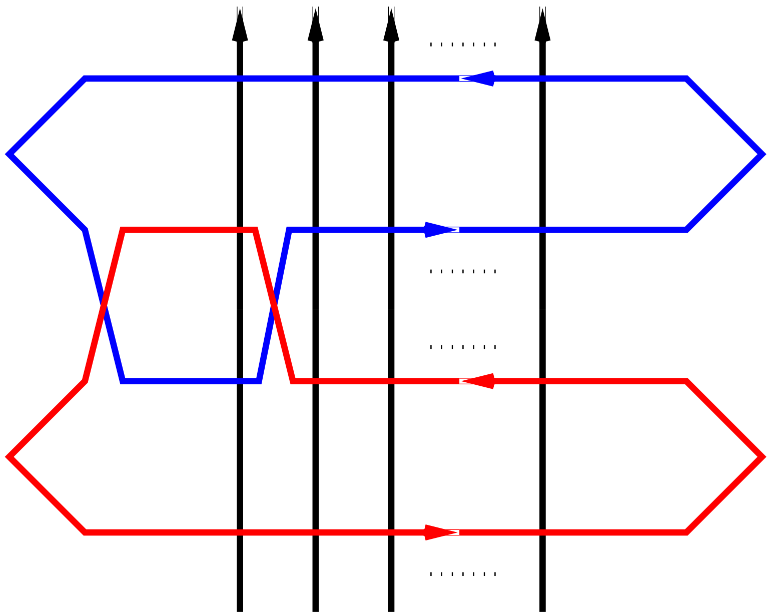
$t(\lambda_1)\ t(\lambda_2) =$



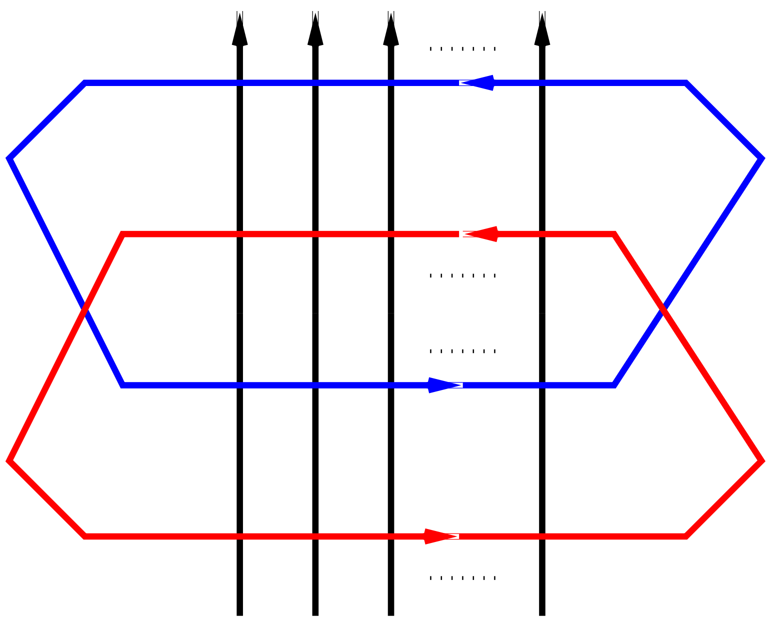
Insertion of crossing unitarity:



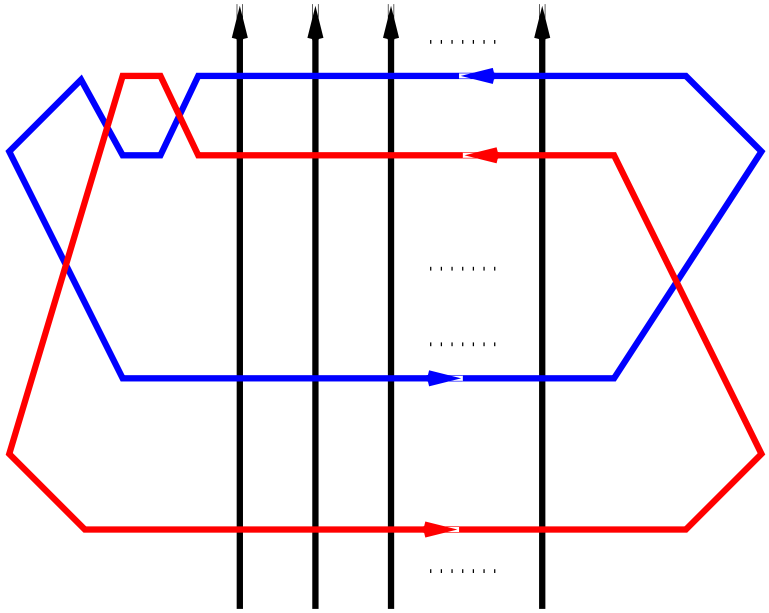
Use of Yang–Baxter equation:



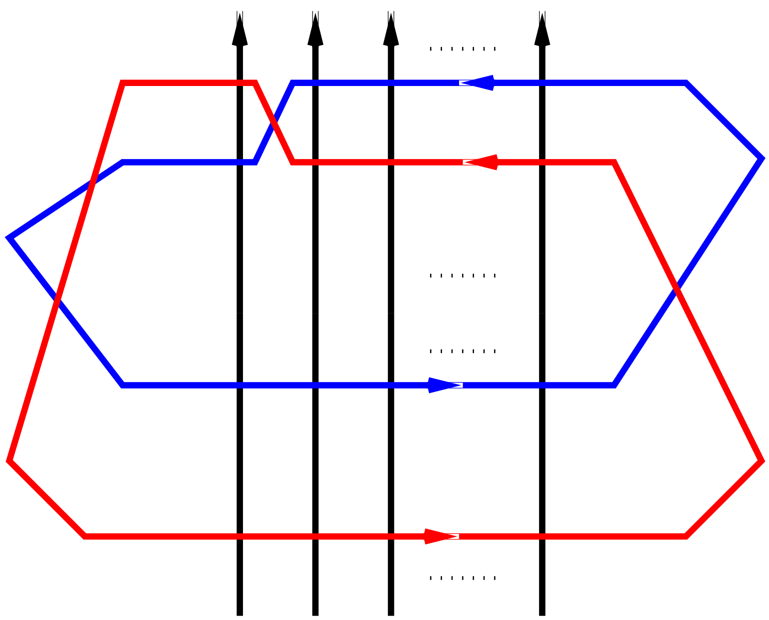
again and again:



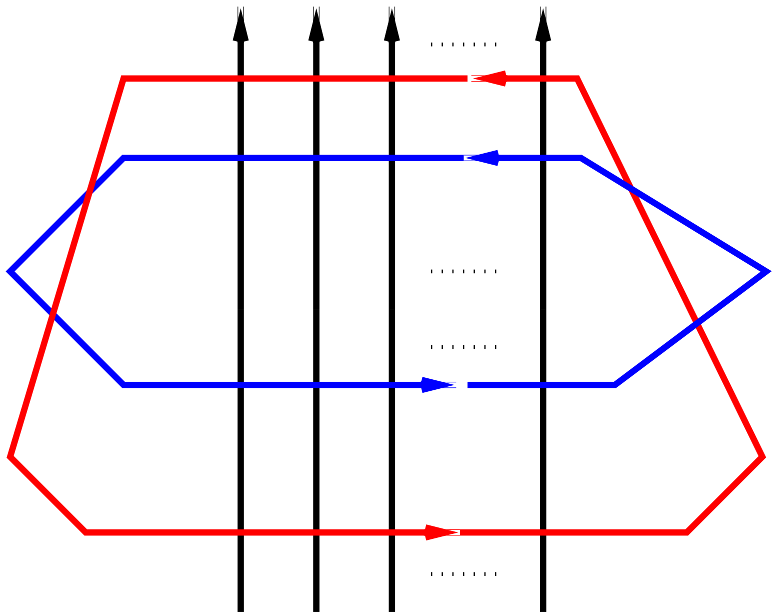
Insertion of $R\ R^{-1}$:



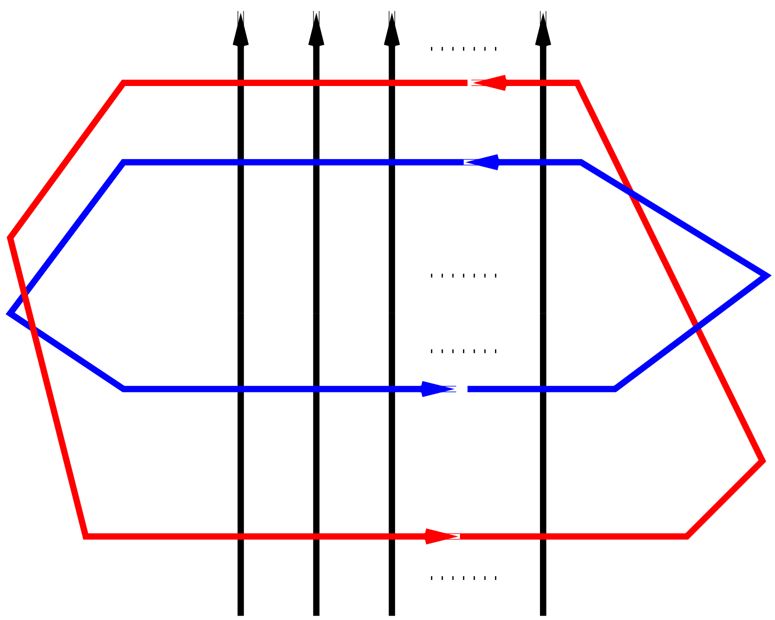
Use of Yang–Baxter equation:



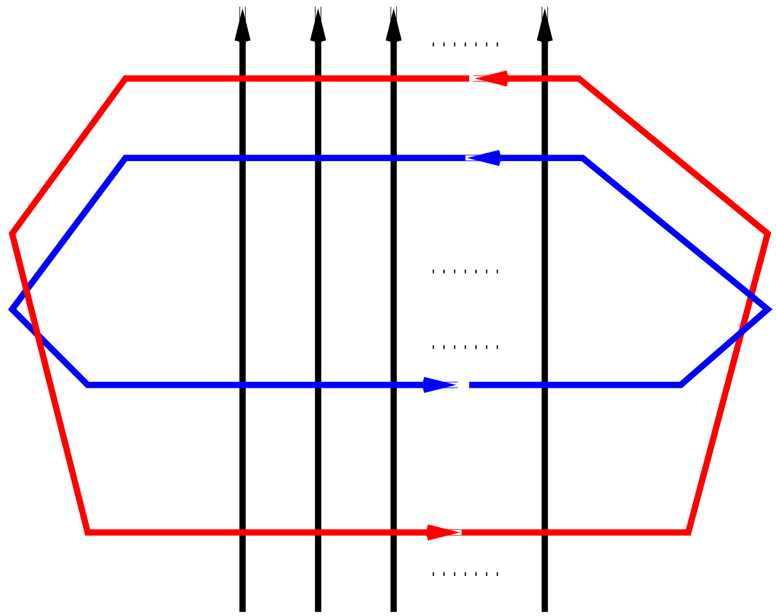
Use of Yang–Baxter again:



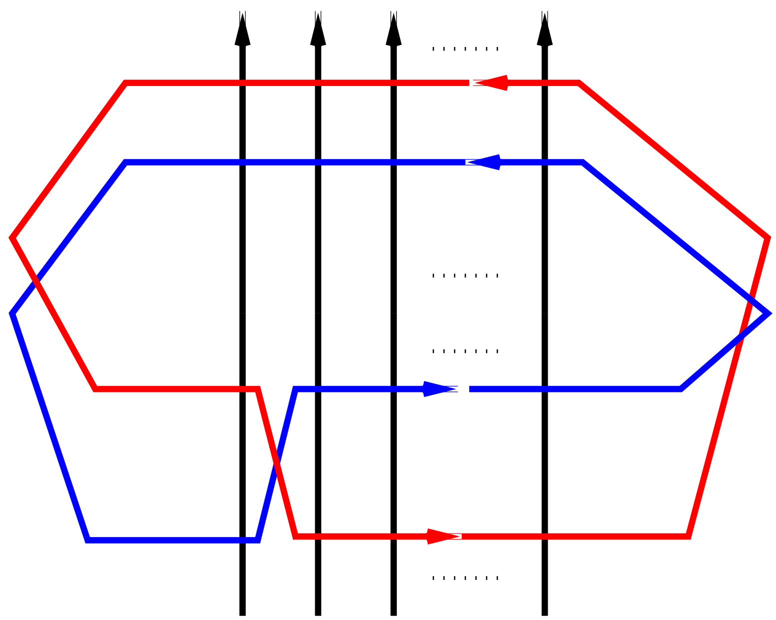
Use of reflection equation on the left:



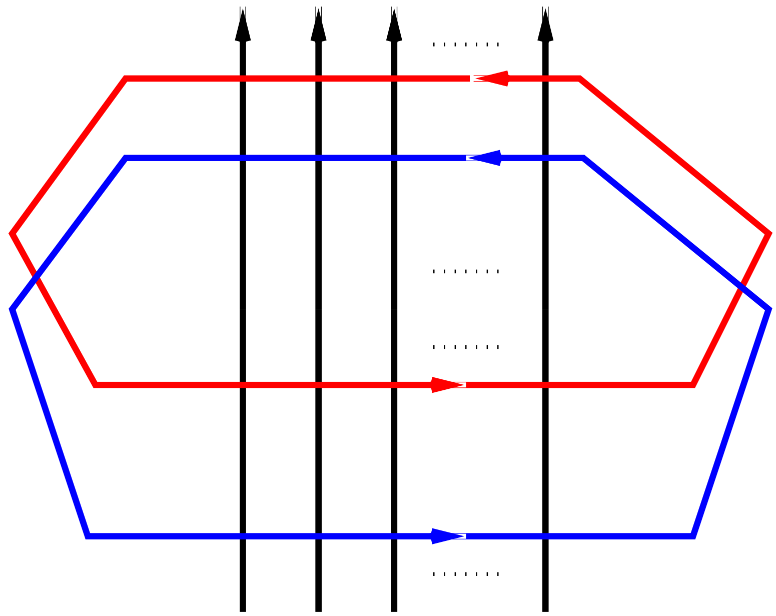
Use of reflection equation on the right:



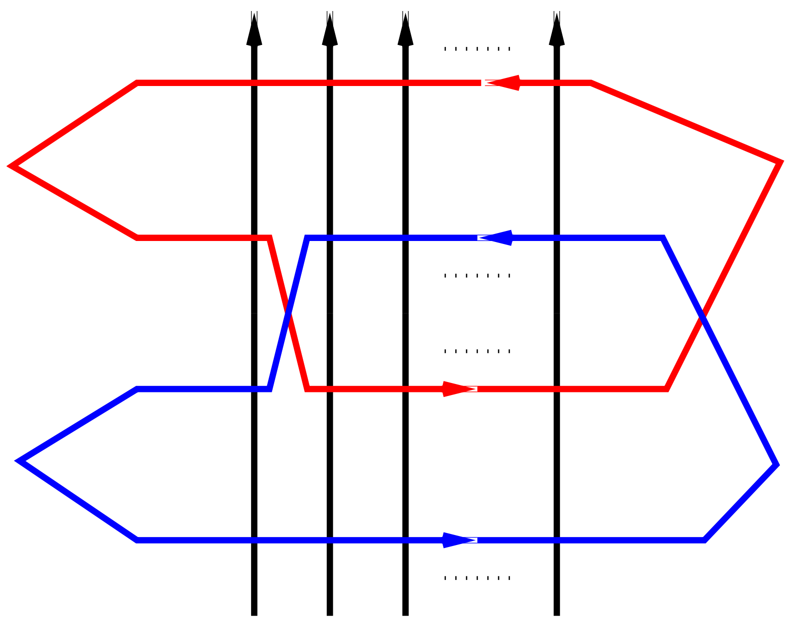
Use of Yang–Baxter equation:



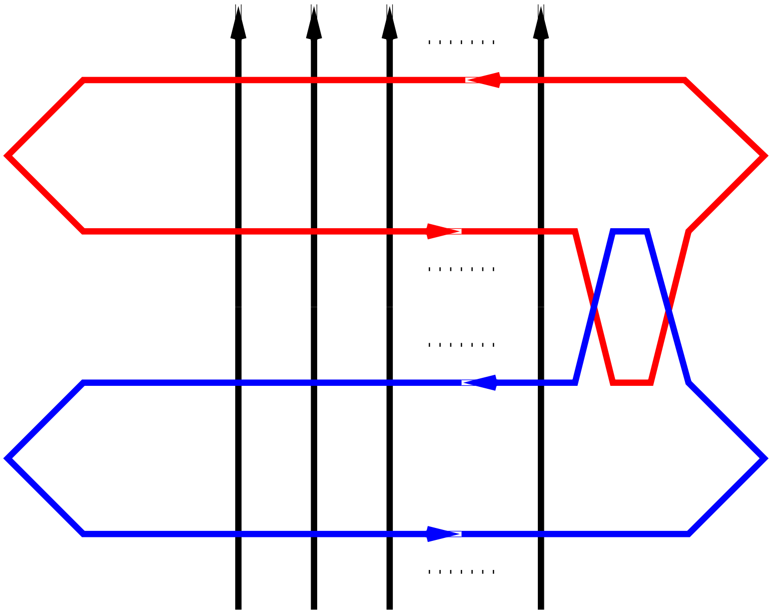
Use of Yang–Baxter again:



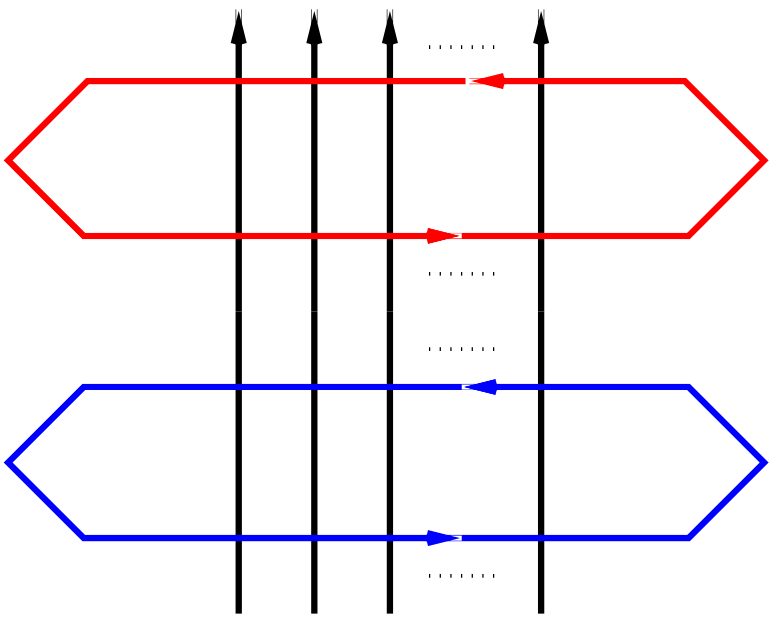
Another Yang–Baxter:



Yang–Baxter up to the right end:



Use of crossing unitarity:

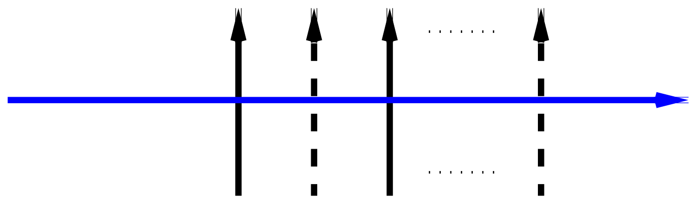


which is $t(\lambda_2)\,t(\lambda_1)$. Hence, $[t(\lambda_1), t(\lambda_2)] = 0$.

4 Soliton non-preserving case

In this section, we consider the case where the reflection on the boundary of the open chain also exchanges the fundamental representation and its conjugate (soliton non preserving case). The monodromy matrix itself is changed and includes alternating fundamental–conjugate vector spaces along the chain (which is supposed to have an even length $2L$).

$$T_a(\lambda) = R_{a\,2L}(\lambda)\bar{R}_{a\,2L-1}(\lambda)\dots R_{a\,2}(\lambda)\bar{R}_{a\,1}(\lambda) \tag{11}$$

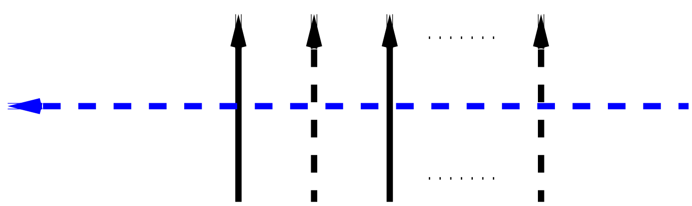


with $\bar{R}(\lambda) = R^{t_1}(-\lambda - \mathrm{i}\rho) = R^{t_2}(-\lambda - \mathrm{i}\rho)$ and $2\rho = \theta_0(M-N)$, $\theta_0 = \pm 1$. We use a transposition ^{t} which is related to the usual transposition ^{T} by (A is any matrix):

$$A^t = V^{-1}\, A^T\, V\,, \quad \text{where} \quad \begin{cases} V = \text{antidiag}(1,1,\dots,1)\,, \\ \qquad \text{for which} \;\; V^2 = \theta_0 = 1 \\ \text{or} \\ V = \text{antidiag}\Big(\underbrace{1,\dots,1}_{N/2}, \underbrace{-1,\dots,-1}_{N/2}\Big)\,, \\ \qquad \text{for which} \;\; V^2 = \theta_0 = -1\,. \end{cases} \tag{12}$$

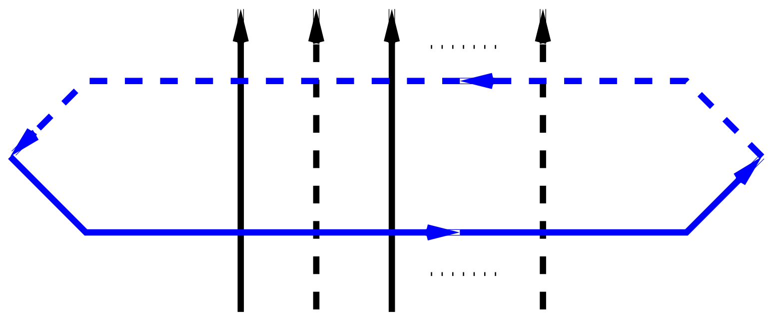
The second case is forbidden for N odd.

$$\hat{T}_{\bar{a}}(\lambda) = R_{1\,a}(\lambda)\bar{R}_{2\,a}(\lambda)\dots R_{2L-1\,a}(\lambda)\bar{R}_{2L\,a}(\lambda)\,. \tag{13}$$



The two–line transfer matrix for the open chain with soliton non-preserving boundary conditions is then defined by

$$t(\lambda) = \mathrm{Tr}_a\, \tilde{K}^+_a(\lambda)\; T_a(\lambda)\tilde{K}^-_a(\lambda)\hat{T}_{\bar{a}}(\lambda)\,, \tag{14}$$



where Tr_a denotes here the *super* trace over the auxiliary space.

The commutation of transfer matrices for different values of the spectral parameter now relies on the (local) reflection equation

$$R_{ab}(\lambda_a-\lambda_b)\tilde{K}_a(\lambda_a)\bar{R}_{ba}(\lambda_a+\lambda_b)\tilde{K}_b(\lambda_b) = \tilde{K}_b(\lambda_b)\bar{R}_{ab}(\lambda_a+\lambda_b)\tilde{K}_a(\lambda_a)R_{ba}(\lambda_a-\lambda_b)\,. \tag{15}$$

5 Solutions of the reflection equation

5.1 Solutions to the soliton−preserving reflection equation

Any bosonic invertible solution of the soliton preserving reflection equation (RE)

$$\begin{aligned} &R_{12}(\lambda_1-\lambda_2)K_1(\lambda_1)R_{12}(\lambda_1+\lambda_2)K_2(\lambda_2) = \\ &\qquad = K_2(\lambda_2)R_{12}(\lambda_1+\lambda_2)K_1(\lambda_1)R_{12}(\lambda_1-\lambda_2)\,, \end{aligned} \tag{16}$$

where $R_{12}(\lambda) = \lambda\,\mathbb{I} + \mathrm{i}\,P_{12}$ is the super–Yangian R–matrix, is of the form

$$K(\lambda) = U\ (i\xi\,\mathbb{I} + \lambda\,\mathbb{E})\,U^{-1}\,, \tag{17}$$

where U is independent of λ and either

(i) $\mathbb{E}$ is diagonal and $\mathbb{E}^2 = \mathbb{I}$ (diagonalisable solutions)

(ii) $\mathbb{E}$ is strictly triangular and $\mathbb{E}^2 = 0$ (non-diagonalisable solutions)

5.2 Solutions to the soliton NON-preserving reflection equation

Any bosonic invertible solution of the soliton non-preserving RE

$$\begin{aligned} &R_{12}(\lambda_1-\lambda_2)\tilde{K}_1(\lambda_1)R_{21}^{t_1}(\lambda_1+\lambda_2)\tilde{K}_2(\lambda_2) = \\ &\qquad = \tilde{K}_2(\lambda_2)R_{12}^{t_1}(\lambda_1+\lambda_2)\tilde{K}_1(\lambda_1)R_{21}(\lambda_1-\lambda_2)\,, \end{aligned} \tag{18}$$

where $R_{12}(\lambda) = \lambda\,\mathbb{I} + \mathrm{i}\,P_{12}$ is the super–Yangian R–matrix, is a constant matrix (up to a multiplication by a scalar function) such that $\tilde{K}^t = \pm\tilde{K}$.

6 Pseudovacuum and one eigenvalue of the transfer matrix

We now choose an appropriate pseudo–vacuum, which is an exact eigenstate of the transfer matrix:

$$|\omega_+\rangle = \bigotimes_{i=1}^{L} |+\rangle_i \quad \text{where} \quad |+\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{M+N}, \tag{19}$$

i.e.

$$t(\lambda) \mid \omega_+\rangle = \Lambda^0(\lambda) \mid\omega_+\rangle \tag{20}$$

with

$$\begin{aligned} \Lambda^0(\lambda) &= \alpha(\lambda)^L g_0(\lambda) + \beta(\lambda)^L \sum_{l=1}^{M+N-2} (-1)^{[l+1]} g_l + (\lambda) \\ &+\gamma(\lambda)^L (-1)^{[M+N-1]} g_{M+N-1}(\lambda)\,, \end{aligned} \tag{21}$$

where, using

$$\begin{aligned} a(\lambda) &= \lambda + \mathrm{i}\,, & b(\lambda) &= \lambda\,, \\ \bar{a}(\lambda) &= a(-\lambda - \mathrm{i}\rho)\,, & \bar{b}(\lambda) &= b(-\lambda - \mathrm{i}\rho) \end{aligned} \tag{22}$$

the functions α, β, γ and g_l are defined as:

(i) Soliton preserving boundary conditions with L sites

$$\alpha(\lambda) = a^2(\lambda)\,, \quad \beta(\lambda) = \gamma(\lambda) = b^2(\lambda) \tag{23}$$

and

$$\begin{aligned} g_l(\lambda) &= \frac{\lambda(\lambda + \frac{\mathrm{i}(M-N)}{2})}{(\lambda + \frac{\mathrm{i}l}{2})(\lambda + \frac{\mathrm{i}(l+1)}{2})}\,, & l &= 0, \ldots, M-1\,, \\ g_l(\lambda) &= \frac{\lambda(\lambda + \frac{\mathrm{i}(M-N)}{2})}{(\lambda + \frac{\mathrm{i}(2M-l-1)}{2})(\lambda + \frac{\mathrm{i}(2M-l)}{2})}\,, & l &\geq M\,. \end{aligned} \tag{24}$$

(ii) Soliton non-preserving boundary conditions with $2L$ sites
The basis used until now was the distinguished basis of $sl(M|N)$, where the indices of $sl(M)$ come in first place, $1, \cdots, M$ and those of $sl(N)$ afterwards $M+1, \cdots, M+N$. In the soliton non preserving case we take $N = 2n$ even. We consider in that case the symmetric basis for $sl(M|2n)$, where the $2n$ indices of $sl(2n)$ are split in two parts: $1, \cdots, n$ and $n+M+1, \cdots, n+M+n$, whereas the indices of the $sl(M)$ part are in the middle: $n+1, \cdots, M+n$. (see [1] for details)

$$\alpha(\lambda) = \Big(a(\lambda)\bar{b}(\lambda)\Big)^2, \quad \beta(\lambda) = \Big(b(\lambda)\bar{b}(\lambda)\Big)^2, \quad \gamma(\lambda) = \Big(\bar{a}(\lambda)b(\lambda)\Big)^2 \tag{25}$$

and

$$\begin{aligned} g_l(\lambda) &= \frac{\lambda + \frac{\mathrm{i}}{2}(\rho - 1)}{\lambda + \frac{\mathrm{i}\rho}{2}}\,, & 0 &\leq l < \frac{M+N-1}{2}\,, \\ g_{\frac{M+N-1}{2}}(\lambda) &= 1\,, & &\text{if } M+N \quad \text{odd}\,, \\ g_l(\lambda) &= g_{N+M-l-1}(-\lambda - \mathrm{i}\rho)\,. \end{aligned} \tag{26}$$

7 Analytical Bethe ansatz

The other eigenvalues are supposed to be obtained by "dressing" with rational functions

$$\begin{aligned}\Lambda(\lambda) = \alpha(\lambda)^L g_0(\lambda) A_0(\lambda) + \beta(\lambda)^L \sum_{l=1}^{M+N-2} (-1)^{[l+1]} g_l(\lambda) A_l(\lambda) + \\ + \gamma(\lambda)^L (-1)^{[M+N-1]} g_{M+N-1}(\lambda) A_{M+N-1}(\lambda)\,.\end{aligned} \tag{27}$$

7.1 Bethe ansatz equations in the soliton preserving case

From the analyticity of $\Lambda(\lambda)$, one gets

$$\begin{aligned} A_l\left(-\frac{\mathrm{i}l}{2}\right) &= A_{l-1}\left(-\frac{\mathrm{i}l}{2}\right), \qquad && l = 1, \dots, M-1\,, \\ A_{2M-l}\left(-\frac{\mathrm{i}l}{2}\right) &= A_{2M-l-1}\left(-\frac{\mathrm{i}l}{2}\right), \qquad && l = M-N+1, \dots, M-1\,. \end{aligned} \tag{28}$$

Gathering together all the constraints one can determine the dressing functions, i.e.

$$\begin{aligned} A_0(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda + \lambda_j^{(1)} - \frac{\mathrm{i}}{2}}{\lambda + \lambda_j^{(1)} + \frac{\mathrm{i}}{2}} \, \frac{\lambda - \lambda_j^{(1)} - \frac{\mathrm{i}}{2}}{\lambda - \lambda_j^{(1)} + \frac{\mathrm{i}}{2}}\,, \\ A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda + \lambda_j^{(l)} + \frac{\mathrm{i}l}{2} + \mathrm{i}}{\lambda + \lambda_j^{(l)} + \frac{\mathrm{i}l}{2}} \, \frac{\lambda - \lambda_j^{(l)} + \frac{\mathrm{i}l}{2} + \mathrm{i}}{\lambda - \lambda_j^{(l)} + \frac{\mathrm{i}l}{2}} \times \\ &\quad \times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda + \lambda_j^{(l+1)} + \frac{\mathrm{i}l}{2} - \frac{\mathrm{i}}{2}}{\lambda + \lambda_j^{(l+1)} + \frac{\mathrm{i}l}{2} + \frac{\mathrm{i}}{2}} \, \frac{\lambda - \lambda_j^{(l+1)} + \frac{\mathrm{i}l}{2} - \frac{\mathrm{i}}{2}}{\lambda - \lambda_j^{(l+1)} + \frac{\mathrm{i}l}{2} + \frac{\mathrm{i}}{2}}\,, \quad l = 1, \dots, M-1\,, \\ A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda + \lambda_j^{(l)} + \mathrm{i}M - \frac{\mathrm{i}l}{2} - \mathrm{i}}{\lambda + \lambda_j^{(l)} + \mathrm{i}M - \frac{\mathrm{i}l}{2}} \, \frac{\lambda - \lambda_j^{(l)} + \mathrm{i}M - \frac{\mathrm{i}l}{2} - \mathrm{i}}{\lambda - \lambda_j^{(l)} + \mathrm{i}M - \frac{\mathrm{i}l}{2}} \times \\ &\quad \times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda + \lambda_j^{(l+1)} + \mathrm{i}M - \frac{\mathrm{i}l}{2} + \frac{\mathrm{i}}{2}}{\lambda + \lambda_j^{(l+1)} + \mathrm{i}M - \frac{\mathrm{i}l}{2} - \frac{\mathrm{i}}{2}} \, \frac{\lambda - \lambda_j^{(l+1)} + \mathrm{i}M - \frac{\mathrm{i}l}{2} + \frac{\mathrm{i}}{2}}{\lambda - \lambda_j^{(l+1)} + \mathrm{i}M - \frac{\mathrm{i}l}{2} - \frac{\mathrm{i}}{2}}\,, \\ &\quad l = M, \dots, M+N-1\,. \end{aligned} \tag{29}$$

Analyticity around the poles introduced in the factors A_l finally imposes the so-called Bethe equations in the λ_i:

$$
\begin{aligned}
e_1(\lambda_i^{(1)})^{2L} &= -\prod_{j=1}^{M^{(1)}} e_2(\lambda_i^{(1)}-\lambda_j^{(1)})\; e_2(\lambda_i^{(1)}+\lambda_j^{(1)}) \times \\
&\quad \times \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)}-\lambda_j^{(2)})\; e_{-1}(\lambda_i^{(1)}+\lambda_j^{(2)})\,, \\
1 &= -\prod_{j=1}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\; e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \times \\
&\quad \times \prod_{\tau=\pm 1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\; e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)})\,, \\
&\quad l=2,\dots,M-1,M+1,\dots,M+N-2\,, \\
1 &= \prod_{j=1}^{M^{(M-1)}} e_{-1}(\lambda_i^{(M)}-\lambda_j^{(M-1)})\; e_{-1}(\lambda_i^{(M)}+\lambda_j^{(M-1)}) \times \\
&\quad \times \prod_{j=1}^{M^{(M+1)}} e_{1}(\lambda_i^{(M)}-\lambda_j^{(M+1)})\; e_{1}(\lambda_i^{(M)}+\lambda_j^{(M+1)})\,, \\
1 &= -\prod_{j=1}^{M^{(M+N-2)}} e_{-1}(\lambda_i^{(M+N-1)}-\lambda_j^{(M+N-2)})\; e_{-1}(\lambda_i^{(M+N-1)}+\lambda_j^{(M+N-2)}) \times \\
&\quad \times \prod_{j=1}^{M^{(M+N-1)}} e_{2}(\lambda_i^{(M+N-1)}-\lambda_j^{(M+N-1)})\; e_{2}(\lambda_i^{(M+N-1)}+\lambda_j^{(M+N-1)})
\end{aligned}
\tag{30}
$$

with

$$
e_x(\lambda) = \frac{\lambda + \frac{\mathrm{i}x}{2}}{\lambda - \frac{\mathrm{i}x}{2}}\,. \tag{31}
$$

We now implement non trivial soliton preserving boundary conditions K^-. From the classification given in section 5, we know that $K^-(\lambda)$ is always conjugated (by a constant matrix U) to a diagonal matrix of the form

$$
K(\lambda) = \mathrm{diag}(\underbrace{\alpha,\dots,\alpha}_{m_1},\underbrace{\beta,\dots,\beta}_{m_2},\underbrace{\beta,\dots,\beta}_{n_2},\underbrace{\alpha,\dots,\alpha}_{n_1})\,. \tag{32}
$$

Then, it is easy to see that the spectrum and the symmetry of the model depend only on the diagonal, and not on U. Indeed, when considering two reflection matrices related by a constant conjugation, the corresponding transfer matrices are also conjugated. Thus, it is enough to consider diagonal $K^-(\lambda)$ matrices to get the general case. Such a property, which relies on the form of the R–matrix, is a priori

valid only in the *rational* $sl(N)$ and $sl(M|N)$ cases.
For a diagonal solution with $m_1 + m_2 = M$, $n_1 + n_2 = N$, $\alpha(\lambda) = -\lambda + \mathrm{i}\xi$, $\beta(\lambda) = \lambda + \mathrm{i}\xi$, and the free boundary parameter ξ, one can compute the new form $\widetilde{g}_l(\lambda)$ of the g–functions entering the expression of $\widetilde{\Lambda}_0(\lambda)$, the new pseudo–vacuum eigenvalue. They take the form:

$$\begin{aligned} &\widetilde{g}_l(\lambda) = (-\lambda + \mathrm{i}\xi)\, g_l(\lambda)\,, && l = 0, \dots, m_1 - 1\,, \\ &\widetilde{g}_l(\lambda) = (\lambda + \mathrm{i}\xi + \mathrm{i}m_1)\, g_l(\lambda)\,, && l = m_1, \dots, M + n_2 - 1\,, \\ &\widetilde{g}_l(\lambda) = (-\lambda + \mathrm{i}\xi - \mathrm{i}m_2 + \mathrm{i}n_2)\, g_l(\lambda)\,, && l = M + n_2, \dots, M + N - 1\,, \end{aligned} \tag{33}$$

where $g_l(\lambda)$ are given by (24). The dressing functions A_l keep the same form, but the Bethe ansatz equations are modified (by $K^-(\lambda)$), so that the value of the eigenvalues $\Lambda(\lambda)$ are different from the ones obtained when $K(\lambda) = \mathbb{I}$.

The modifications induced on Bethe ansatz equations are the following:
– The factor $-e^{-1}_{2\xi+m_1}(\lambda)$ appears in the LHS of the ${m_1}^{\text{th}}$ Bethe equation.
– The factor $-e^{-1}_{2\xi+m_1-m_2-n_2}(\lambda)$ appears in the LHS of the $(M + n_2)^{\text{th}}$ Bethe equation.

7.2 Bethe ansatz equations in the soliton non preserving case

The dressing functions now take the form:

$$\begin{aligned} A_0(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda + \lambda_j^{(1)} - \frac{\mathrm{i}}{2}}{\lambda + \lambda_j^{(1)} + \frac{\mathrm{i}}{2}}\, \frac{\lambda - \lambda_j^{(1)} - \frac{\mathrm{i}}{2}}{\lambda - \lambda_j^{(1)} + \frac{\mathrm{i}}{2}}\,, \\ A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda + \lambda_j^{(l)} + \frac{\mathrm{i}l}{2} + \mathrm{i}}{\lambda + \lambda_j^{(l)} + \frac{\mathrm{i}l}{2}}\, \frac{\lambda - \lambda_j^{(l)} + \frac{\mathrm{i}l}{2} + \mathrm{i}}{\lambda - \lambda_j^{(l)} + \frac{\mathrm{i}l}{2}} \times \\ &\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda + \lambda_j^{(l+1)} + \frac{\mathrm{i}l}{2} - \frac{\mathrm{i}}{2}}{\lambda + \lambda_j^{(l+1)} + \frac{\mathrm{i}l}{2} + \frac{\mathrm{i}}{2}}\, \frac{\lambda - \lambda_j^{(l+1)} + \frac{\mathrm{i}l}{2} - \frac{\mathrm{i}}{2}}{\lambda - \lambda_j^{(l+1)} + \frac{\mathrm{i}l}{2} + \frac{\mathrm{i}}{2}}\,, \qquad l = 1, \dots, n-1\,, \\ A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda + \lambda_j^{(l)} + \mathrm{i}n - \frac{\mathrm{i}l}{2} - \mathrm{i}}{\lambda + \lambda_j^{(l)} + \mathrm{i}n - \frac{\mathrm{i}l}{2}}\, \frac{\lambda - \lambda_j^{(l)} + \mathrm{i}n - \frac{\mathrm{i}l}{2} - \mathrm{i}}{\lambda - \lambda_j^{(l)} + \mathrm{i}n - \frac{\mathrm{i}l}{2}} \times \\ &\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda + \lambda_j^{(l+1)} + \mathrm{i}n - \frac{\mathrm{i}l}{2} + \frac{\mathrm{i}}{2}}{\lambda + \lambda_j^{(l+1)} + \mathrm{i}n - \frac{\mathrm{i}l}{2} - \frac{\mathrm{i}}{2}}\, \frac{\lambda - \lambda_j^{(l+1)} + \mathrm{i}n - \frac{\mathrm{i}l}{2} + \frac{\mathrm{i}}{2}}{\lambda - \lambda_j^{(l+1)} + \mathrm{i}n - \frac{\mathrm{i}l}{2} - \frac{\mathrm{i}}{2}}\,, \\ &\qquad n \le l < n + \frac{M-1}{2} \end{aligned} \tag{34}$$

and $A_l(\lambda) = A_{M+2n-1-l}(-\lambda - i\rho)$, and for $M = 2m+1$

$$A_k(\lambda) = \prod_{j=1}^{M^{(k)}} \frac{\lambda + \lambda_j^{(k)} + \mathrm{i}n - \frac{\mathrm{i}k}{2} - \mathrm{i}}{\lambda + \lambda_j^{(k)} + \mathrm{i}n - \frac{\mathrm{i}k}{2}} \, \frac{\lambda - \lambda_j^{(k)} + \mathrm{i}n - \frac{\mathrm{i}k}{2} - \mathrm{i}}{\lambda - \lambda_j^{(k)} + \mathrm{i}n - \frac{\mathrm{i}k}{2}} \times$$
$$\times \frac{\lambda + \lambda_j^{(k)} + \mathrm{i}n - \frac{\mathrm{i}k}{2} + \frac{\mathrm{i}}{2}}{\lambda + \lambda_j^{(k)} + \mathrm{i}n - \frac{\mathrm{i}k}{2} - \frac{\mathrm{i}}{2}} \, \frac{\lambda - \lambda_j^{(k)} + \mathrm{i}n - \frac{\mathrm{i}k}{2} + \frac{\mathrm{i}}{2}}{\lambda - \lambda_j^{(k)} + \mathrm{i}n - \frac{\mathrm{i}k}{2} - \frac{\mathrm{i}}{2}}\,, \qquad (k = m+n) \tag{35}$$

and the Bethe ansatz equations read as:

A. $\mathbf{sl(2m+1|2n)}$ superalgebra

$$e_1(\lambda_i^{(1)})^{2L} = -\prod_{j=1}^{M^{(1)}} e_2(\lambda_i^{(1)} - \lambda_j^{(1)})\; e_2(\lambda_i^{(1)} + \lambda_j^{(1)}) \times$$
$$\times \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)} - \lambda_j^{(2)})\; e_{-1}(\lambda_i^{(1)} + \lambda_j^{(2)})\,,$$
$$1 = -\prod_{j=1}^{M^{(l)}} e_2(\lambda_i^{(l)} - \lambda_j^{(l)})\; e_2(\lambda_i^{(l)} + \lambda_j^{(l)}) \times$$
$$\times \prod_{\tau=\pm 1} \prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)} - \lambda_j^{(l+\tau)})\; e_{-1}(\lambda_i^{(l)} + \lambda_j^{(l+\tau)})\,,$$
$$l = 2, \ldots, n+m-1\,, \quad l \neq n\,,$$
$$1 = \prod_{j=1}^{M^{(n+1)}} e_1(\lambda_i^{(n)} - \lambda_j^{(n+1)})\; e_1(\lambda_i^{(n)} + \lambda_j^{(n+1)}) \times$$
$$\times \prod_{j=1}^{M^{(n-1)}} e_{-1}(\lambda_i^{(n)} - \lambda_j^{(n-1)})\; e_{-1}(\lambda_i^{(n)} + \lambda_j^{(n-1)})\,,$$
$$e_{-\frac{1}{2}}(\lambda_i^{(k)}) = -\prod_{j=1}^{M^{(k)}} e_2(\lambda_i^{(k)} - \lambda_j^{(k)})\; e_2(\lambda_i^{(k)} + \lambda_j^{(k)})\; e_{-1}(\lambda_i^{(k)} - \lambda_j^{(k)})\; e_{-1}(\lambda_i^{(k)} + \lambda_j^{(k)}) \times$$
$$\times \prod_{j=1}^{M^{(k-1)}} e_{-1}(\lambda_i^{(k)} - \lambda_j^{(k-1)})\; e_{-1}(\lambda_i^{(k)} + \lambda_j^{(k-1)})\,, \quad k = m+n\,. \tag{36}$$

Note that these equations are the Bethe ansatz equations of the $osp(2m+1|2n)$ case (see e.g. [4]) apart from the last equation.

B. **sl(2m|2n)** superalgebra

The first $n+m-1$ equations are the same as in the previous case, but the last equation is modified, with again $k=m+n$, to

$$e_1(\lambda_i^{(k)}) = -\prod_{j=1}^{M^{(k)}} e_2(\lambda_i^{(k)}-\lambda_j^{(k)})\, e_2(\lambda_i^{(k)}+\lambda_j^{(k)}) \times \\ \times \prod_{j=1}^{M^{(k-1)}} e_{-1}^2(\lambda_i^{(k)}-\lambda_j^{(k-1)})\, e_{-1}^2(\lambda_i^{(k)}+\lambda_j^{(k-1)})\,. \qquad (37)$$

With a non trivial diagonal reflection matrix K^-

In the case of a non trivial diagonal reflection matrix K^- with $\varepsilon=1$, i.e.

$$\tilde{K}^-(\lambda) = \mathrm{diag}(k_1,\dots,k_{M+N}) \quad \text{with} \quad k_{M+N+1-j} = k_j\,, \qquad (38)$$

the g–functions entering the new pseudo–vacuum eigenvalue are modified as:

$$\widetilde{g}_l(\lambda) = k_{l+1}\, g_l(\lambda)\,, \quad 0 \le l \le \frac{M+N-1}{2}\,, \qquad (39)$$

where $g_l(\lambda)$ are given by (26). The remaining $\widetilde{g}$ are defined by requiring the crossing relation

$$\widetilde{g}_{M+N-l}(-\lambda-i\rho) = \widetilde{g}_l(\lambda)\,. \qquad (40)$$

The dressing functions (34) and (36) keep the same form, but the LHS of ℓ^{th} Bethe ansatz equation (given in (36) and (37)) is multiplied by $k_\ell/k_{\ell+1}$.

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