Non−generic symmetries and surface terms

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Integrable geometries were obtained by adding a total time derivative involving the components of the angular momentum to a given free Lagrangian. The motion on a sphere and its induced geometries are examined in details.

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1 Introduction

Killing–Yano tensors (KY) were introduced by Yano [1] from pure mathematical point of view [2] and the physical significance of these tensors was obtained by Gibbons and Holten [3]. A KY is an antisymmetric tensor define as

$$D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0\,, \tag{1}$$

where D_λ represents the covariant derivative. KY tensors of rank two are related to non-generic supersymmetries of the spinning particle model (see for more details Ref. [3]) and the geometrical duality depends on the existence of these tensors [4, 5]. Since KY tensors were introduced there were many attempts to applied them in various areas [6, 7, 8, 9, 10].

In this paper we made a link between the surface terms [11] and KY tensors and we review the results presented in [13].

The starting point is a given free Lagrangian $L(\dot q^i, q^i)$ admitting a set of constants of motion denoted by L_i, $i = 1, \cdots, 3$. If we add the components of the angular momentum corresponding to L, the extended Lagrangian [12]

$$L' = L + \dot\lambda^i L_i\,, \quad i = 1, \cdots, 3 \tag{2}$$

becomes $L' = \frac{1}{2} a_{ij}\dot q^i \dot q^j$. In this context the second term in (2) is a total time derivative and the Lagrangians L and L' are equivalent.We mention that the matrix a_{ij} is symmetric by construction. The next step is to find whether a_{ij} is singular or not. Assuming that a_{ij} is a singular $n \times n$ matrix of rank $n-1$ we obtain non-singular symmetric matrices of order $(n-1)\times(n-1)$, where n will be 3, 5 and 6. Finally we consider the obtained matrices as metrics on the extended space and we investigate their Killing vectors and KY tensors.

2 Angular momentum and Killing–Yano tensors

The Lagrangian to start with is

$$L' = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \dot{\lambda}_3\left(x\dot{y} - y\dot{x}\right), \tag{3}$$

which in the compact notation becomes $L' = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$. Here a_{ij} is given by

$$a_{ij} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{pmatrix}. \tag{4}$$

The metric (4) admits the Killing vector $V = (y, -x, 0)$.

Solving (1) for (4) we obtained the following KY tensor

$$f_{12} = 0\,, \quad f_{23} = -Cx\sqrt{x^2 + y^2}\,, \quad f_{13} = Cy\sqrt{x^2 + y^2}\,, \tag{5}$$

where C represents a constant [13].

As it known a KY tensor of rank two generates a Killing tensor as

$$K_{\mu\nu} = f_{\mu\lambda}f^{\lambda}_{\nu}\,. \tag{6}$$

In our case, using (5) and (6) a Killing tensor is constructed as

$$K_{ij} = \begin{pmatrix} y^2 & -xy & -y(y^2+x^2) \\ -xy & x^2 & x(x^2+y^2) \\ -y(y^2+x^2) & x(x^2+y^2) & 0 \end{pmatrix}. \tag{7}$$

The second step is to add two components of the angular momentum to a free, three–dimensional Lagrangian. The corresponding extended Lagrangian becomes

$$L' = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \dot{\lambda}_1\left(y\dot{z} - z\dot{y}\right) + \dot{\lambda}_2\left(z\dot{x} - x\dot{z}\right) \tag{8}$$

and from (8) we obtain a_{ij} as the following non-singular matrix

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 1 & y & -x \\ 0 & -z & y & 0 & 0 \\ z & 0 & -x & 0 & 0 \end{pmatrix}. \tag{9}$$

The metric (9) admits three Killing vectors as

$$V_1 = (y, -x, 0, 0, 0)\,, \quad V_2 = (0, -z, y, 0, 0)\,, \quad V_3 = (z, 0, -x, 0, 0)\,. \tag{10}$$

For metric (10) KY tensors components are as follows

$$\begin{aligned} f_{15} &= -Gxy\,, & f_{14} &= G(z^2 + y^2)\,, \\ f_{24} &= -Gxy\,, & f_{34} &= -Gxz, \\ f_{25} &= G(x^2 + z^2)\,, & f_{35} &= \frac{-Gxzy}{x}\,, \\ f_{12} &= Cz\,, & f_{13} &= -Cy\,, \end{aligned} \tag{11}$$

others zero. Here C and G are constants. The corresponding Killing tensor has the following form

$$K = \begin{pmatrix} G(-2C+G)(z^2+y^2) & GDxy & GDzx & 0 & G^2r^2z \\ GDxy & -GD(x^2+z^2) & GDzy & -r^2zG^2 & 0 \\ GDzx & GDzy & -GD(y^2+x^2) & G^2r^2y & -G^2r^2x \\ 0 & -G^2zr^2 & G^2yr^2 & 0 & 0 \\ G^2zr^2 & 0 & -G^2xr^2 & 0 & 0 \end{pmatrix}. \quad (12)$$

where $D = 2C + G$ and $r^2 = x^2 + y^2 + z^2$.

If we add all angular momentum components to the Lagrangian of the free particle in three–dimensions, the extended Lagrangians L' is given by

$$L' = \tfrac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \dot{\lambda}_1\left(y\dot{z} - z\dot{y}\right) + \dot{\lambda}_2\left(z\dot{x} - x\dot{z}\right) + \dot{\lambda}_3\left(x\dot{y} - y\dot{x}\right). \quad (13)$$

In compact form (13) has the form $L' = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$. Here a_{ij} is singular matrix given by

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \\ 0 & -z & y & 0 & 0 & 0 \\ z & 0 & -x & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

Using the fact that the rank of (14) is 5 we obtained three non-singular symmetric matrices corresponding to three non-zero minors. The first one is given by (9) and the other two are as

$$b^{(2)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & -y \\ 0 & 1 & 0 & -z & x \\ 0 & 0 & 1 & y & 0 \\ 0 & -z & y & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix} \quad (15)$$

and

$$b^{(3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & z & -y \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & -x & 0 \\ z & 0 & -x & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}. \quad (16)$$

By direct calculations [13] we obtain that (15) and (16) admit three Killing vectors given by (10) and a KY tensor possessing the following non-zero components

$$f_{12} = z\,, \quad f_{13} = -y\,, \quad f_{23} = x\,. \quad (17)$$

3 Induced geometries on a sphere

The motion on a sphere admits four constants of motion, the Hamiltonian and three components of the angular momentum [14]. The aim of this section is to use the surface terms and to generate four–dimensional manifolds. The Lagrangian to start with is given by

$$L' = \frac{1}{2}\left(1+\frac{x^2}{u}\right)\dot{x}^2 + \frac{1}{2}\left(1+\frac{y^2}{u}\right)\dot{y}^2 + \frac{xy}{u}\,\dot{x}\dot{y} - \frac{xy}{\sqrt{u}}\,\dot{\lambda}_1\dot{x} + \left(\frac{x^2}{\sqrt{u}}+\sqrt{u}\right)\dot{\lambda}_2\dot{x} - \\ - \left(\frac{y^2}{\sqrt{u}}+\sqrt{u}\right)\dot{\lambda}_1\dot{y} + \frac{xy}{\sqrt{u}}\,\dot{\lambda}_2\dot{y} + x\dot{\lambda}_3\dot{y} - y\dot{\lambda}_3\dot{x}\,, \tag{18}$$

where $u = 1 - x^2 - y^2$. Using (18) we identify the singular matrix a_{ij} as

$$a_{ij} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}}+\sqrt{u} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & \frac{xy}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & 0 & 0 & 0 \\ \frac{x^2}{\sqrt{u}}+\sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

Using the fact that (19) is a singular matrix of rank 4 we identify three symmetric minors of order four. If we consider these minors as a metric we observed that they are not conformaly flat but their scalar curvatures are zero.

The first metric is given by

$$g^{(1)}_{\mu\nu} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & \sqrt{u}+\frac{x^2}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & \frac{xy}{\sqrt{u}} & x \\ \sqrt{u}+\frac{x^2}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix}. \tag{20}$$

The Killing vectors of (20) are given by [13]

$$\begin{aligned} V_1 &= (y, -x, 0, 0)\,, \\ V_2 &= \left(\sqrt{1-x^2-y^2}+\frac{x^2}{1-x^2-y^2}, \frac{xy}{1-x^2-y^2}, 0, 0\right), \\ V_3 &= \left(-\frac{xy}{1-x^2-y^2}, -\sqrt{1-x^2-y^2}-\frac{y^2}{1-x^2-y^2}, 0, 0\right). \end{aligned} \tag{21}$$

The next step is to investigate its KY tensors. Solving (1) we obtain the following set of solutions:

a. One solution is $f_{21} = \dfrac{C_1}{\sqrt{1-x^2-y^2}}$, others zero.

b. Two-by-two solution has the form: $f_{31} = f_{42} = C$.

c. Three by three solution is $f_{21} = \dfrac{C_1}{\sqrt{-1+x^2+y^2}}$ and $f_{31} = f_{42} = C$, where C and C_1 are constants.

From (18) another two metrics can be identified as

$$g^{(2)}_{\mu\nu} = \begin{pmatrix} 1+\dfrac{x^2}{u} & \dfrac{xy}{u} & -\dfrac{xy}{\sqrt{u}} & -y \\ \dfrac{xy}{u} & 1+\dfrac{y^2}{u} & -\sqrt{u}-\dfrac{y^2}{\sqrt{u}} & x \\ -\dfrac{xy}{\sqrt{u}} & -\sqrt{u}-\dfrac{y^2}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix} \tag{22}$$

and

$$g^{(3)}_{\mu\rho} = \begin{pmatrix} 1+\dfrac{x^2}{u} & \dfrac{xy}{u} & -\dfrac{xy}{\sqrt{u}} & \dfrac{x^2}{\sqrt{u}}+\sqrt{u} \\ \dfrac{xy}{u} & 1+\dfrac{y^2}{u} & -\dfrac{y^2}{\sqrt{u}}-\sqrt{u} & \dfrac{xy}{\sqrt{u}} \\ -\dfrac{xy}{\sqrt{u}} & -\dfrac{y^2}{\sqrt{u}}-\sqrt{u} & 0 & 0 \\ \dfrac{x^2}{\sqrt{u}}+\sqrt{u} & \dfrac{xy}{\sqrt{u}} & 0 & 0 \end{pmatrix}. \tag{23}$$

By direct calculations we obtained that (22) and (23) have the same Killing vector as in (21). Solving (1) for (22) and (23) we find one non-zero component of KY tensor as follows

$$f_{21} = \frac{C_1}{\sqrt{1-x^2-y^2}}\,. \tag{24}$$

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References

[1] K. Yano: Ann. Math. **55** (1952) 328.

[2] D. Kramer, H. Stephani, E. Herlt and M. MacCallum: *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, Cambridge, 1980.

[3] G.W. Gibbons, R.H. Rietdijk and J.W. van Holten: Nucl. Phys. B **404** (1993) 42.

[4] R.H. Rietdijk and J.W. van Holten: Nuc. Phys. B **472** (1996) 427.

[5] F. Hinterleitner: Ann. Phys. **271** (1999) 23.

[6] B.Carter and R.G. McLenaghan: Phys. Rev. D **19** (1979) 1093.

[7] C.D. Collinson: Int. J. Theor. Phys. **15** (1976) 311.

[8] B. Carter and R.G. McLenaghan: Phys. Rev. D **19** (1979) 1093.

[9] R. Penrose: Ann. NY Acad. Sci. **224** (1973) 125;
R. Floys: *The dynamics of Kerr fields*, Ph. D. Thesis, London, 1973.

[10] D. Baleanu: Nuovo Cimento B **114** (1999) 1065.

[11] M. Henneaux, C. Teitelboim and J.D. Vergara: Nucl. Phys. B **387** (1991) 391.

[12] Y. Güler, D. Baleanu and M. Cenk: Nuovo Cimento B **118** (2003) 293.

[13] D.Baleanu, Ö.Defterli: Czech. Journ. Phys. **54** (2004) 157.

[14] T.L. Curtright and C.K. Zachos: New J. Phys. **4** (2002) 83.