Little group kinematics associated with classical optics

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Little groups are the subgroups of the Poincaré group whose transformations leave the four–momentum of a relativistic particle invariant. Massless particle representations can be obtained from their massive counterparts through a contraction procedure. The two by two matrix representations of little groups as well as of other kinematical effects of special relativity, like Wigner rotations are observed to coincide with the matrix formulations of some interesting classical ray optics phenomena. Examples include beam cycles in laser cavities, image focusing in a one–lens–camera, multilens optics and interferometers. Thus, it is argued that optical implementations can be exploited as analogue processors for special relativity.

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1 Introduction

The mathematical formulation of special relativity is based on the Lorentz group. The Lorentz group contains six parameters. However it is possible to obtain the most general form of a Lorentz transformation starting from two rotation and one boost transformation matrices, which is due to the fact that the third rotation can be obtained by successive applications of the rotations, and boosts can be aligned in any direction by rotations. The two by two matrix representations of this group belongs to $SL(2, C)$.

It is by now a well established fact that classical ray optics can be formulated in terms of two by two matrices with determinant one [1]. The basic matrix elements are the translation and the lens (or mirror) matrices. They contain the optical parameters as the distance "d", and the focal length "f" (or the curvature of the mirror "R"), respectively.

In this exposition, we shall first associate these two optical parameters with the parameters of the Lorentz group, keeping in mind that there is an implicit rotation around the optical axis of the system. We will then give some optical phenomena such as lens focusing or cavity stability which can mathematically be formulated in the same manner as the contractions of little groups [2] or Wigner rotations, [3] which are known to be the topics of special relativity.

Therefore, it is in order to argue that having shared the same mathematical formulation, one can deduce relativistic effects by performing optical experiments in laboratories, where such systems are much more economical and accessible compared to experiments designed for relativistic effects.

2 Little group kinematics

2.1 Little groups and contractions

Little groups are the maximal subgroups of the Poincaré group whose transformations leave the four–momentum of a relativistic particle invariant, while changing the direction of its spin [4]. The little group for a massive particle at rest is the O(3) group. The little group for a relativistic particle in motion is isomorphic to the rotation group. Namely the transformations become boosted rotations. On the other hand for massless particles there are no Lorentz frames in which the particle is at rest. The best way is to align its spin in one particular direction.

The little group for a massive particle in motion can be obtained from boosted rotation generators as

$$J_i' = B^{-1} J_i B\,, \quad i = 1,\, 2,\, 3\,. \tag{1}$$

and are isomorphic to $O(3)$.

The little group for massless particles can be obtained through the contraction procedure. It is the infinite momentum limit of the $O(3)$–like little group for relativistic particles in motion. More explicitly we have

$$N_1 = \lim_{\eta\to\infty} \mathrm{e}^{-\eta} B^{-1} J_2 B\,, \quad N_2 = \lim_{\eta\to\infty} -\mathrm{e}^{-\eta} B^{-1} J_1 B\,, \tag{2}$$

where η is the boost parameter. The generators N_1 and N_2 are the contracted J_2 and J_1 respectively in the infinite–momentum limit. They satisfy the commutation relations

$$[N_1, N_2] = 0\,, \quad [J_3, N_1] = \mathrm{i}N_2\,, \quad [J_3, N_2] = -\mathrm{i}N_1 \tag{3}$$

of the Euclidean group. Thus, the little group for massless particles is the cylindrical group which is isomorphic to the $E(2)$ group [5].

Table 1. Little groups for relativistic particles

Particle	rest/motion	Little group
massive	rest	$O(3)$
massive	motion	$O(3)$–like
massless	motion	$E(2)$–like

Roughly, the contraction procedure of a mathematical object can be described as first transforming, then normalizing and finally taking the infinite limit of the transformation parameter. To illustrate this let us consider the four–momentum $p^\mu = (1, 0, 0, 0)$ of a massive particle at rest. Throughout the text the four vector convention will be adopted as (E, p_z, p_x, p_y). Then with the boost matrix

$$\begin{pmatrix} \sqrt{p^2+1} & p & 0 & 0 \\ p & \sqrt{p^2+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

the four–momentum is boosted to be in motion, which then becomes

$$p^\mu = (\sqrt{p^2+1}, p, 0, 0)\,, \tag{5}$$

where the group parameter η is related to the transformation parameter p as

$$\mathrm{e}^{-\eta} = \left(\frac{\sqrt{p^2+1} - p^2}{\sqrt{p^2+1} + p^2} \right)^{1/2} . \tag{6}$$

Finally, the four–momentum is normalized and its infinite momentum limit is taken

$$\lim_{p\to\infty} \left[\frac{1}{p} \left(\sqrt{p^2+1}, p, 0, 0 \right) \right] = (1, 1, 0, 0) \tag{7}$$

to yield the four–momentum of a massless particle.

2.2 Rotations associated with Lorentz boosts

It is possible to associate two angles with two successive non-colinear Lorentz boosts. If one boost is applied after the initial boost, the result is a final boost proceeded or followed by a rotation, called the Wigner rotation. The other rotation is associated with Wigner's $O(3)$–like little group. Apparently these two angles are different, yet the sum of these two angles is equal to a third angle, which is the angle between the initial and the final boost.

More explicitly the Wigner rotation angle w can be expressed as

$$R(w) = R(\theta) B(0, -\eta) R(-\theta) B(\psi, \lambda) B(0, \eta)\,, \tag{8}$$

where $B(\psi, \lambda)$ represents a boost transformation which makes an angle ψ from the z axis, while $R(\theta)$ stands for the rotation around the y axis. The little group transformation which leaves the four–momentum

$$p^\mu = (\cosh\eta, \sinh\eta, 0, 0) \tag{9}$$

invariant is $B(0, \eta) R(\alpha) B(0, -\eta)$, where α is the Wigner's little group angle. Furthermore another little group transformation can be performed on (9), and these two can be equated:

$$B(\psi, -\lambda) R(\theta) = B(0, \eta) R(\alpha) B(0, -\eta)\,. \tag{10}$$

The above equation is valid when applied to the four vector in (9), while the expression (8) for Wigner rotation is a mathematical identity. In view of (10), (8) can be rewritten as

$$B(\psi, -\lambda)R(\theta) = B(0, \eta)R(\theta - \omega)B(0, -\eta)\,, \tag{11}$$

which leads to

$$\alpha = \theta - \omega \tag{12}$$

meaning that the Lorentz angle θ between the initial and the final boost is the sum of the little group angle α and the Wigner rotation angle ω [6].

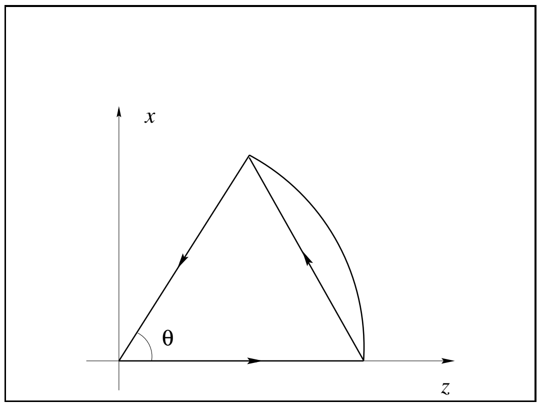

Fig. 1. Three successive boosts yield to a Wigner rotation of Eq.(8). The little group kinematics of this figure consists of a rotation by an angle of θ and the inverse of the second boost which is given in the left hand side of Eq. (10).

The little group angle α is found in terms of the Lorentz group parameters as

$$\tan\alpha = \frac{\sin\theta\, \cosh\eta}{\cos\theta\, \cosh^2\eta + (\cosh^2\eta - 1)\, \sin^2(\theta/2)}\,. \tag{13}$$

3 Optical systems

In this section we shall consider some optical systems where their defining two-by-two matrices share the same mathematical formulation with that of the special relativity through the two by two representations of the Lorentz group.

3.1 Camera−like one−lens system and contraction of little groups

In analyzing optical rays in para–axial lens optics, we start with the lens matrix and the translation matrix written as

$$L = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & d \\ -0 & 1 \end{pmatrix}, \tag{14}$$

respectively. Then the one–lens system consists of

$$\begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - d_2/f & d_1 + d_2 - d_1 d_2/f \\ -1/f & 1 - d_1/f \end{pmatrix} . \tag{15}$$

The focal condition

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f} , \tag{16}$$

can be obtained if we impose that the upper–right element of (15) is to be zero. Then, the right hand side of (15) can be decomposed as

$$\begin{pmatrix} (d_1 d_2)^{1/4} & 0 \\ 0 & (d_1 d_2)^{1/4} \end{pmatrix} \begin{pmatrix} 1 - x_2 & 2\cosh\rho - x \\ -x & 1 - x_1 \end{pmatrix} \begin{pmatrix} (d_1 d_2)^{1/4} & 0 \\ 0 & (d_1 d_2)^{1/4} \end{pmatrix} , \tag{17}$$

where

$$x_i = \frac{d_i}{f} , \quad x = \frac{\sqrt{d_1 d_2}}{f} , \quad \cosh\rho = \frac{1}{2} \left(\sqrt{\frac{d_1}{d_2}} + \sqrt{\frac{d_2}{d_1}} \right) . \tag{18}$$

Our main purpose is to associate little groups and the group contraction procedure to the focusing of the one lens camera [7]. The following reparametrization of the core matrix of the above decomposition will prove to be useful

$$C = \begin{pmatrix} z - 1 & x - 2\cosh\rho \\ x & z - 1 \end{pmatrix} , \tag{19}$$

where

$$z = 1 + \sqrt{x^2 - 2\cosh\rho + 1} . \tag{20}$$

During the focusing process the upper right corner of (19) can be negative, then can continuously be made zero to focus the image, and focuses away while it continuously becomes positive. For each of these three cases one can associate a transformation matrix belonging to the little group of timelike, null and spacelike particles with a suitable reparametrization.

Case i. If $x < 2\cosh\rho$, then the upper right corner of (19) is negative and the reparametrization is chosen as

$$\begin{pmatrix} \cos\phi & -\mathrm{e}^{-\eta}\sin\phi \\ \mathrm{e}^{\eta}\sin\phi & \cos\phi \end{pmatrix} . \tag{21}$$

This matrix is an $O(3)$–like boosted rotation which leaves the four–momentum $p^\mu = (\cosh\eta, -\sinh\eta, 0, 0)$ invariant, of a massive particle moving in $-z$ direction.

Case ii. If $x = 2\cosh\rho$, then the core matrix is reparatmetrized as

$$\begin{pmatrix} 1 & 0 \\ 2\cosh\rho & 1 \end{pmatrix} , \tag{22}$$

which is $E(2)$–like, and leaves the momentum $p^\mu = (1, 1, 0, 0)$ of a massless particle invariant.

Case iii. If $x > 2\cosh\rho$, then

$$\begin{pmatrix} \cosh(\xi/2) & -e^{-\eta}\sinh(\xi/2) \\ e^{\eta}\sinh(\xi/2) & \cosh(\xi/2) \end{pmatrix}, \tag{23}$$

which is $O(2,1)$–like and leaves the momentum $p^{\mu} = (0,1,0,0)$ of a space–like particle invariant.

Both $O(3)$–like of (21) and $O(2,1)$–like of (23) can be contracted to become $E(2)$–like of (22). However, this is a singular transformation and the reverse transformation is not unique. In order to circumvent this singularity problem, another little group configuration with different set of parameters can be chosen. For this purpose we first rotate by an angle θ the four-momentum

$$p^{\mu} = (\cosh\eta, -\sinh\eta, 0, 0) \tag{24}$$

moving along the $-z$ direction with the speed of $\tanh\eta$. Then it can be boosted along the x direction, and returns to its original value by again rotating it with the same angle. Then the overall effect of this transformation is the same as (21), which leads us to equate

$$\begin{pmatrix} z-1 & x-2\cosh\rho \\ x & z-1 \end{pmatrix} = \begin{pmatrix} \cosh\lambda\cos\theta & -\cosh\lambda\sin\theta + \sin\lambda \\ \cosh\lambda\sin\theta + \sin\lambda & \cosh\lambda\cos\theta \end{pmatrix}. \tag{25}$$

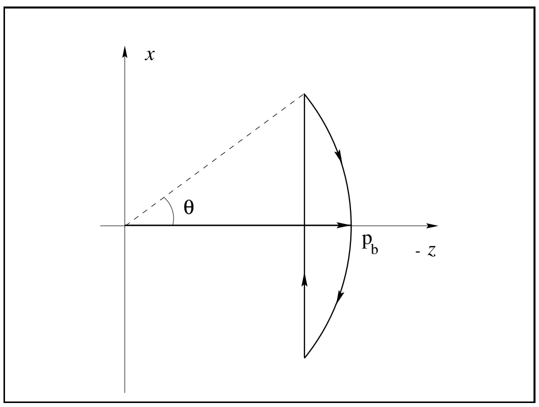

Fig. 2. The little group kinematics of the right hand side of Eq.(25) which consists of a rotation by an angle θ, boost along the x direction and and a final rotation, leaving the four–momentum of (24) invariant.

The optical parameters are related to the little group parameters in the following way:

$$x - 2\cosh\rho = \sin\lambda - \cosh\lambda\sin\theta\,, \tag{26}$$

$$x = \sin\lambda + \cosh\lambda\sin\theta\,, \tag{27}$$

$$\sin\lambda = x - \cosh\rho \tag{28}$$

and

$$\sin\theta = \frac{\cosh\rho}{[1+(x-\cosh\rho)^2]^{1/2}}\,. \tag{29}$$

Therefore, changing the little group parameters from ϕ and η to θ and λ it is possible to navigate analytically through the vanishing value of the upper right element of (19). The process of approaching this zero value either from the positive or negative side is called the group contraction.

3.2 Multilens systems

We consider a co-axial system of an arbitrary number of lenses, where the focal lengths and the separation between lenses are not the same. Such a system is expressed as

$$T_1L_1T_2L_2T_3L_3\ldots T_NL_N\,, \tag{30}$$

which results in a general $ABCD$ matrix. The translation and the lens matrices are generated by

$$\mathrm{e}^{-\mathrm{i}dX_1}, \quad \mathrm{e}^{\mathrm{i}X_2/f}, \tag{31}$$

where

$$X_1 = \begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ \mathrm{i} & 0 \end{pmatrix}. \tag{32}$$

Then we have

$$[X_1, X_2] = -\mathrm{i}X_3\,, \tag{33}$$

with

$$X_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{34}$$

These generators satisfy the following commutation relations

$$[X_1, X_3] = -2\mathrm{i}X_1\,, \quad [X_2, X_3] = 2\mathrm{i}X_2\,, \tag{35}$$

which are recognized as $Sp(2)$. The most general form of an $ABCD$ matrix can be decomposed as [8]:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\eta} & 0 \\ 0 & \mathrm{e}^{-\eta} \end{pmatrix} \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \tag{36}$$

with $AD - CB = 1$. The expression above can further take the form

$$\begin{aligned} &\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\eta} & 0 \\ 0 & \mathrm{e}^{-\eta} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \\ &= \begin{pmatrix} \cosh\eta + \sinh\eta\cos 2\phi & \sinh\eta\sin 2\phi \\ \sinh\eta\sin 2\phi & \cosh\eta - \sinh\eta\cos 2\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \end{aligned} \tag{37}$$

which is expressed as the product of one symmetric and one rotation matrix, i.e., $(ABCD) = SR$. The rotation and the symmetric matrices can be written as

$$R = \begin{pmatrix} 0 & -\tan(\theta/2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin\theta & 1 \end{pmatrix} \begin{pmatrix} 0 & -\tan(\theta/2) \\ 0 & 1 \end{pmatrix} \tag{38}$$

and

$$S = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}, \tag{39}$$

where a and b, which contain the optical parameters are related to the group parameters as

$$\begin{aligned} a &= \pm\left[(\cosh\eta - 1) + \sinh\eta\cos 2\phi\right]^{1/2} \\ b &= \frac{\sinh\eta\sin 2\phi \pm \left[(\cosh\eta - 1) + \sinh\eta\cos 2\phi\right]^{1/2}}{\cosh\eta + \sinh\eta\cos 2\phi} . \end{aligned} \tag{40}$$

Thus the $ABCD$ matrix becomes

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b - \tan(\theta/2) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin\theta & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tan(\theta/2) \\ 0 & 1 \end{pmatrix} . \tag{41}$$

Therefore, we conclude that the most general form an $ABCD$ matrix contains minimum three lens matrices which are appropriately separated by three translation matrices [9].

3.3 Laser cavities

We consider a simple laser cavity consisting of two identical concave mirrors separated by a distance d. Then the $ABCD$ matrix for one round trip of a beam is

$$\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \tag{42}$$

where R is the radius of the mirror, and the cycle of the beam starts from one of the mirrors. On the other hand one can start the beam at the midpoint between the mirrors, in order to analyze the condition when the beam cycle repeats itself for many times. Then the $ABCD$ matrix can be expressed as a similarity transformation

$$AC^2A^{-1}, \tag{43}$$

or more explicitly as

$$\begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix} \begin{pmatrix} 1 - d/R & 1 - d^2/2R \\ -2d/R & 1 - d/R \end{pmatrix}^2 \begin{pmatrix} 1/\sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}, \tag{44}$$

where C in the middle is the core matrix. We are now able to associate a little group transformation of a massive particle in motion as in (21) to C as

$$C = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix}, \tag{45}$$

where

$$\cos\phi = 1 - \frac{d}{R}, \quad e^{2\eta} = \frac{R}{2d} - \frac{1}{4} \tag{46}$$

are the relations between the little group parameters and the optical parameters.

If the process is repeated N times, (43) becomes

$$AC^{2N}A^{-1}, \tag{47}$$

then we have

$$C^{2N} = \begin{pmatrix} \cos(2N\phi) & -e^{\eta}\sin(2N\phi) \\ e^{\eta}\sin(2N\phi) & \cos(2N\phi) \end{pmatrix}. \tag{48}$$

The Lorentz group angle θ is calculated as

$$\theta = 2\tan^{-1}\left(\sin(\phi/2)\sqrt{\cosh\eta}\right). \tag{49}$$

Through (46) the laser cavity presents two parameters η and ϕ. So forth we can now calculate the Wigner rotation angle ω from $\omega = \theta - \phi$ given in (12). Thus, one Wigner rotation corresponds to the beam going through one cycle in the laser cavity [10].

4 Conclusion

We have shown in this exposition that the Lorentz group embraces two seemingly different branches of physics, namely special relativity and classical optics.

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