Nonlinear algebraic structures *)

N. Belaloui†), L. Khodja‡) and H. Bennacer

LPMPS, Département de Physique, Faculté des Sciences,
Université Mentouri Constantine,
Constantine, Algeria

We paraquantize the bosonic (resp. the Neuveu Shwarz spinning) string theory. Unlike the Ardalan and Mansouri work, the paraquantum system is so that both the center of mass variables and the excitation modes of the string verify paracommutation relations. We find existence possibilities of parabosonic (resp. paraspinning) strings defined in a noncommutative space–time at space–time dimensions other than $D = 26$ (resp. $D = 10$). We investigate then the existence possibilities of the. $D = 3$, 4, 6 parasuperstring. The two cases, parabose–parafermi (resp. bose–parafermi) superstrings are considered. In the first one, the spectrum is discussed through the partition functions for $D = 3$, 4, 6. Despite of the parastatistical algebraic structure of the dynamical variables, the combined set of the generators of the symmetries forms the algebra of the Super Symmetric Quantum Mechanic (resp. the ParaSSQM in the sense of Beckers and Debergh).

PACS: 11.30.Cp , 11.25.Hf
Key words: parastring, Poincar parasuperalgebra, critical dimentions, noncommutativity

1 Introduction

A first study of the Poincaré algebra in the parabosonic (resp. paraspinning) string theories was done by F.Ardalan and F.Mansouri [1]. This study is based on the particular manner in which the center of mass variables of the string are to be handled. Indeed, these authors impose on the center of mass coordinates and the total energy momentum operators of the string x^μ, p^μ to satisfy ordinary commutation relations. This is done by the choice of a specific direction in the paraspace of the Green components, characterized by the anzatz $x^{\mu(\beta)} = x^\mu \delta_{\beta,1}$ and $p^{\mu(\beta)} = p^\mu \delta_{\beta,1}$, where $x^{\mu(\beta)}$ (resp. $p^{\mu(\beta)}$) are the Green components of x^μ (resp. p^μ). This requires relative paracommutation relations between the center of mass coordinates and the excitation modes of the string which are exclusively anomalous bilinear commutation relations in terms of the Green components. Because of the separation of $\beta = 1$ and $\beta \neq 1$ in the precedent anzatz, these bilinear commutations relations can not be rewritten in trilinear commutation relations form which are the basis of the paraquantization. In this hypothesis, they find that the resulting parabosonic (resp. paraspinning) string theories are Poincaré invariant if the dimension D of the space–time and the order Q of the paraquantization are related by the expressions $D = 2 + 24/Q$ (resp. $D = 2 + 8/Q$). In this work, paraquantizing this theory consists in reinterpreting the classical bosonic and fermionic string

*) Presented at the XI–th International Conference Symmetry Methods in Physics, Prague, 21–24 June 2004.
†) E-mail: n.belaloui@wissal.dz
‡) E-mail: lamine.khodja@caramail.com

variables $X^{\mu}(\sigma,\tau)$, $\mathcal{P}^{\nu}(\sigma',\tau)$, and $\psi^{\rho}(\sigma'',\tau)$ as operators satisfying the paraquantum trilinear commutation relations (τ is a time like evolution parameter, while the parameter σ labels points on the string). Unlike Ardalan and Mansouri work, this is done by requiring that both the center of mass variables and the excitation modes of the string verify paraquantum commutation relations.

In the first part of this paper, we review some results of [2], [3], which consists in a critical study of the Poincaré algebra in the cases of the parabosonic and the Neuveu Shwarz paraspinning string theories.

In the section 2 we construct the parabosonic (resp. the paraspinning) string formalism, we discuss the three commutators of the Poincaré algebra, we find that, except the $[p^{\mu},p^{\nu}]$ commutator which conduct to a discussion about the space–time properties, the two others are satisfied. In the section 3 we determine the space–time critical dimensions where, as in the Ardalan and Mansouri work, the relations between the space–time dimensions D and the order of the paraquantization Q are again $D=2+24/Q$ (resp. $D=2+8/Q$). In particular, one can have paraspinning string with the critical dimensions $D=10$, 6, 4, 3 resp. in the orders $Q=1$, 2, 4, 8. This coincide with the dimensions in which the classical superstrings can be formulated. In the second part of this work, we investigate the existence possibilities of the $D=3$, 4, 6 parasuperstring. In connection with some works on the extension of the supersymmetric quantum mechanic (SSQM) to the paraSSQM (see for example [4], [5], [6] and the references herein), our system correspond to a parabose–parafermi supersymmetric system in the sense that despite of the parastatistical algebraic structure of the dynamical variables, the combined set of the generators of the symmetries forms the algebra of the SSQM. In this case, the spectrum is discussed through the partition functions for $D=3$, 4, 6. In connection with some works on the fractional superstrings based on a bosonic–fractional fermionic system which are again formulated in $D=2$, 3, 4, 6, 10 (see for example [7], [8] and the references herein), a boson–parafermi superstrings are considered. Unlike the first case, the combined set of the generators of the symmetries forms the algebra of the PSSQM in the sense of Beckers and Debergh.

2 Covariant gauge

2.1 Paraquantum formalism

2.1.1 Parabosonic case

The paraquantization of the theory is carried out by reinterpreting the classical dynamical variables $X^{\mu}(\sigma,\tau)$ and $P^{\mu}(\sigma,\tau)$ as operators satisfying the so-called trilinear commutation relations

$$\left[X^{\mu}(\sigma,\tau),\left[P^{\nu}(\sigma',\tau),P^{\rho}(\sigma'',\tau)\right]_{+}\right]=2\mathrm{i}g^{\mu\nu}P^{\rho}\delta(\sigma-\sigma')+2\mathrm{i}g^{\mu\rho}P^{\nu}\delta(\sigma-\sigma''),\tag{1}$$

$$\left[P^{\mu}(\sigma,\tau),\left[X^{\nu}(\sigma',\tau),X^{\rho}(\sigma'',\tau)\right]_{+}\right]=-2\mathrm{i}g^{\mu\nu}X^{\rho}\delta(\sigma-\sigma')-2\mathrm{i}g^{\mu\rho}X^{\nu}\delta(\sigma-\sigma''),\tag{2}$$

$$\left[X^{\mu}(\sigma,\tau), [X^{\nu}(\sigma',\tau), P^{\rho}(\sigma'',\tau)]_{+}\right] = 2\mathrm{i}g^{\mu\rho}X^{\nu}\delta(\sigma-\sigma''), \tag{3}$$
$$\left[P^{\mu}(\sigma,\tau), [X^{\nu}(\sigma',\tau), P^{\rho}(\sigma'',\tau)]_{+}\right] = 2\mathrm{i}g^{\mu\nu}P^{\rho}\delta(\sigma-\sigma'). \tag{4}$$

Rewritten in terms of the center of mass variables x^{μ}, p^{μ} and the excitation modes α_n^{μ}, equations (1–4) are equivalent to:

$$\left[x^{\mu}, [x^{\nu}, p^{\rho}]_{+}\right] = 2\mathrm{i}g^{\mu\rho}x^{\nu}, \tag{5}$$
$$\left[x^{\mu}, [p^{\nu}, p^{\rho}]_{+}\right] = 2\mathrm{i}\left(g^{\mu\nu}p^{\rho} + g^{\mu\rho}p^{\nu}\right), \tag{6}$$
$$\left[x^{\mu}, [p^{\nu}, \alpha_n^{\rho}]_{+}\right] = 2\mathrm{i}g^{\mu\nu}\alpha_n^{\rho}, \tag{7}$$
$$\left[p^{\mu}, [x^{\nu}, p^{\rho}]_{+}\right] = -2\mathrm{i}g^{\mu\nu}p^{\rho}, \tag{8}$$
$$\left[p^{\mu}, [x^{\nu}, x^{\rho}]_{+}\right] = -2\mathrm{i}\left(g^{\mu\nu}x^{\rho} + g^{\mu\rho}x^{\nu}\right), \tag{9}$$
$$\left[\alpha_n^{\mu}, [\alpha_m^{\nu}, \alpha_l^{\rho}]_{+}\right] = 2\left(g^{\mu\nu}n\delta_{n+m,0}\alpha_l^{\rho} + g^{\mu\rho}n\delta_{n+l,0}\alpha_m^{\nu}\right), \tag{10}$$
$$\left[\alpha_n^{\mu}, [p^{\nu}, \alpha_m^{\rho}]_{+}\right] = 2ng^{\mu\rho}\delta_{n+m,0}p^{\nu}, \tag{11}$$
$$\left[\alpha_n^{\mu}, [x^{\nu}, \alpha_m^{\rho}]_{+}\right] = 2ng^{\mu\rho}\delta_{n+m,0}x^{\nu} \tag{12}$$

and all the other commutators are null.

In terms of the Green components, the trilinear commutation relations (5–12) transform to the bilinear commutation relations of an anomalous case

$$\left[x^{\mu(\sigma)}, p^{\nu(\sigma)}\right] = \mathrm{i}g^{\mu\nu}, \tag{13}$$
$$\left[x^{\mu(\sigma_1)}, p^{\nu(\sigma_2)}\right]_{+} = 0, \qquad \sigma_1 \neq \sigma_2, \tag{14}$$
$$\left[p^{\mu(\sigma)}, p^{\nu(\sigma)}\right] = \left[x^{\mu(\sigma)}, x^{\nu(\sigma)}\right] = 0, \tag{15}$$
$$\left[p^{\mu(\sigma_1)}, p^{\nu(\sigma_2)}\right]_{+} = \left[x^{\mu(\sigma_1)}, x^{\nu(\sigma_2)}\right]_{+} = 0, \quad \sigma_1 \neq \sigma_2, \tag{16}$$
$$\left[\alpha_n^{\mu(\sigma)}, \alpha_m^{\nu(\sigma)}\right] = ng^{\mu\nu}\delta_{n+m,0}, \tag{17}$$
$$\left[\alpha_n^{\mu(\sigma_1)}, \alpha_l^{\nu(\sigma_2)}\right]_{+} = 0, \qquad \sigma_1 \neq \sigma_2, \tag{18}$$
$$\left[x^{\mu(\sigma)}, \alpha_n^{\nu(\sigma)}\right] = \left[p^{\mu(\sigma)}, \alpha_n^{\nu(\sigma)}\right] = 0, \tag{19}$$
$$\left[x^{\mu(\sigma_1)}, \alpha_n^{\nu(\sigma_2)}\right]_{+} = \left[p^{\mu(\sigma_1)}, \alpha_n^{\nu(\sigma_2)}\right]_{+} = 0, \quad \sigma_1 \neq \sigma_2. \tag{20}$$

2.1.2 Paraspinning case

In the same way, we reinterpret the classical dynamical variables α_n^{μ}, p^{μ}, x^{μ} and b_r^{μ} as operators satisfying the so called trilinear commutation relations:

$$\left[x^{\mu}, [p^{\nu}, A]_{+}\right] = 2\mathrm{i}g^{\mu\nu}A, \tag{21}$$
$$\left[x^{\mu}, [p^{\nu}, p^{\rho}]_{+}\right] = 2\mathrm{i}\left(g^{\mu\nu}p^{\rho} + g^{\mu\rho}p^{\nu}\right), \tag{22}$$
$$\left[\alpha_n^{\mu}, [\alpha_m^{\nu}, \alpha_l^{\rho}]_{+}\right] = 2\left(g^{\mu\nu}n\delta_{n+m,0}\alpha_l^{\rho} + g^{\mu\rho}n\delta_{n+l,0}\alpha_m^{\nu}\right), \tag{23}$$

$$\left[\alpha_n^\mu,[\alpha_m^\nu,B]_+\right] = 2ng^{\mu\nu}\delta_{n+m,0}B\,, \tag{24}$$
$$\left[b_r^\mu,\left[b_s^\nu,b_q^\rho\right]_-\right] = 2\left(g^{\mu\nu}\delta_{r+s,0}b_q^\rho - g^{\mu\rho}\delta_{r+q,0}b_s^\nu\right), \tag{25}$$
$$\left[b_r^\mu,[b_s^\nu,C]_+\right] = 2g^{\mu\nu}\delta_{r+s,0}C \tag{26}$$

and all the other commutators are null. Here $l,n\in\mathbb{Z}$ and $r,s,q\in\left(\mathbb{Z}+\frac{1}{2}\right)$ and A, B, and C represent the following operators:

$$\begin{aligned} A &= \alpha_n^\rho\,,\ x^\rho \ \text{ or } \ b_r^\rho\,, \\ B &= p^\rho\,,\ x^\rho \ \text{ or } \ b_r^\rho\,, \\ C &= p^\rho\,,\ x^\rho \ \text{ or } \ \alpha_n^\rho\,. \end{aligned}$$

2.2 Paraquantum Poincaré algebra

2.2.1 Introduction

We rewrite the generators $M^{\mu\nu}$ basing on an adequate symmetrization which takes the form $M^{\mu\nu}=l^{\mu\nu}+E^{\mu\nu}$ in the parabosonic case with:

$$l^{\mu\nu} = \tfrac{1}{2}\left[x^\mu,p^\nu\right]_+ - \left[x^\nu,p^\mu\right]_+ \tag{27}$$

and

$$E^{\mu\nu} = -\frac{\mathrm{i}}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\left[\alpha_{-n}^\mu,\alpha_n^\nu\right]_+ - \left[\alpha_{-n}^\nu,\alpha_n^\mu\right]_+\right) \tag{28}$$

and in the paraspinning case, takes the form

$$M^{\mu\nu} = M_0^{\mu\nu}(x) + K^{\mu\nu} \tag{29}$$

with

$$M_0^{\mu\nu}(x) = l^{\mu\nu} + E^{\mu\nu} \tag{30}$$

and

$$K^{\mu\nu} = -\frac{\mathrm{i}}{4}\sum_{r=-\infty}^{\infty}\left(\left[b_{-r}^\mu,b_r^\nu\right]_- - \left[b_{-r}^\nu,b_r^\mu\right]_-\right). \tag{31}$$

Now, if this writing allows the elimination of the order ambiguities, it also allows the paraquantum treatment of the problem solely with the trilinear relations (5–12), (21–26) and without having recourse to the Green representation (13–20). One can nevertheless find again the same results with the use of the Green decomposition.

2.2.2 Direct application of the trilinear relations

Let us perform the second and the third commutator of the Poincaré algebra. With the use of the trilinear relations (5–12) for the parabosonic case and (21–26) for the parafermionic one, one can find:

$$\left[p^\mu,M^{\nu\rho}\right] = -\mathrm{i}g^{\mu\nu}p^\rho + \mathrm{i}g^{\mu\rho}p^\nu\,, \tag{32}$$
$$\left[M^{\mu\nu},M^{\rho\sigma}\right] = \mathrm{i}g^{\nu\rho}M^{\sigma\mu} - \mathrm{i}g^{\mu\sigma}M^{\nu\rho} - \mathrm{i}g^{\nu\sigma}M^{\rho\mu} + \mathrm{i}g^{\mu\rho}M^{\nu\sigma}\,, \tag{33}$$

a result which satisfy Poincaré algebra. Now, for the first commutator $[p^\mu, p^\nu]$ of the algebra, one can only write $\left[p^\mu, [p^\nu, p^\sigma]_+\right] = 0$, in the same way for the coordinates x^μ where one can only write $\left[x^\mu, [x^\nu, x^\sigma]_+\right] = 0$. These relations are equivalent to $[p^\mu, p^\nu] = \Theta^{\mu\nu}$ and $[x^\mu, x^\nu] = \Lambda^{\mu\nu}$ where, in terms of the Green components, $\Theta^{\mu\nu} = 2\sum_{\alpha\neq\beta} p^{\mu(\alpha)} p^{\nu(\beta)} \neq 0\,!$ and $\Lambda^{\mu\nu} = 2\sum_{\alpha\neq\beta} x^{\mu(\alpha)} x^{\nu(\beta)} \neq 0\,!$ for $Q \neq 1$. These are noncommuting coordinates and momentum coordinates so that, for $Q \neq 1$, we are working in a noncommutative space–time.

3 Transverse gauge

3.1 Formalism

3.1.1 Parabosonic case

In the same way as before, paraquantizing the theory in this gauge comes down to reinterprete the independent classical dynamical variables x^-, p^+, x^i, p^i and α_n^i as operators satisfying the paracommutation relations:

$$\left[x^i, \left[p^j, p^k\right]_+\right] = 2\mathrm{i}\left(g^{ij} p^k + g^{ik} p^j\right), \tag{34}$$

$$\left[\alpha_n^i, \left[\alpha_m^j, \alpha_l^k\right]_+\right] = 2\left(g^{ij} n \delta_{n+m,0} \alpha_l^k + g^{ik} n \delta_{n+l,0} \alpha_m^j\right), \tag{35}$$

$$\left[x^i, \left[p^j, A\right]_+\right] = 2\mathrm{i}\delta^{ij} A\,, \tag{36}$$

$$\left[x^-, \left[p^+, B\right]_+\right] = 2\mathrm{i}B\,, \tag{37}$$

$$\left[\alpha_n^i, \left[\alpha_m^j, C\right]_+\right] = 2n\delta_{n+m,0}\delta^{ij} C\,, \tag{38}$$

where A, B, and C are given by:

$$\begin{aligned} A &= x^-\,,\ p^+\,,\ x^k\,,\ \text{ or }\ \alpha_n^k\,,\\ B &= x^-\,,\ x^k\,,\ p^k\ \text{ or }\ \alpha_n^k\,,\\ C &= x^-\,,\ p^+\,,\ x^k\ \text{ or }\ p^k\,, \end{aligned}$$

which are equivalent to

$$\left[x^{i(\alpha)}, p^{j(\alpha)}\right] = \mathrm{i}\delta^{ij}\,, \qquad \left[x^{i(\alpha)}, p^{j(\beta)}\right]_+ = 0\,, \quad \alpha \neq \beta\,, \tag{39}$$

$$\left[x^{-(\alpha)}, p^{+(\alpha)}\right] = \mathrm{i}\,, \qquad \left[x^{-(\alpha)}, p^{+(\beta)}\right]_+ = 0\,, \quad \alpha \neq \beta\,, \tag{40}$$

$$\left[\alpha_n^{i(\alpha)}, \alpha_m^{j(\alpha)}\right] = n\delta^{ij}\delta_{n+m,0}\,, \qquad \left[\alpha_n^{i(\alpha)}, \alpha_m^{j(\beta)}\right]_+ = 0\,, \quad \alpha \neq \beta\,, \tag{41}$$

and all the other commutators (and anticommutators) of the type $\left[A^{(\alpha)}, B^{(\alpha)}\right] = 0$ (and $\left[A^{(\alpha)}, B^{(\beta)}\right]_+ = 0$, for $\alpha \neq \beta$).

3.1.2 Pafermionic case

In this gauge, the paraquantum operators x^-, p^+, x^i, p^i, α_n^i and b_r^i verify the trilinear relations:

$$\left[b_r^i, \left[b_s^j, b_q^k\right]_-\right] = 2\left(\delta^{ij}\delta_{r+s}b_q^k - \delta^{ik}\delta_{r+q}b_s^j\right), \tag{42}$$

$$\left[\alpha_n^i, \left[\alpha_m^j, \alpha_l^k\right]_+\right] = 2\left(\delta^{ij}n\delta_{n+m,0}\alpha_l^k + n\delta^{ik}\delta_{n+l,0}\alpha_m^j\right), \tag{43}$$

$$\left[x^i, \left[p^j, pk\right]_+\right] = 2\mathrm{i}\left(\delta^{ij}p^k + \delta^{ik}p^j\right), \tag{44}$$

$$\left[\alpha_n^i, \left[\alpha_m^j, D\right]_+\right] = 2\delta^{ij}n\delta_{n+m}D\,, \tag{45}$$

$$\left[b_r^i, \left[b_s^j, E\right]_+\right] = 2\delta^{ij}\delta_{r+s}E\,, \tag{46}$$

$$\left[x^i, \left[p^j, F\right]_+\right] = 2\mathrm{i}\delta^{ij}F\,, \tag{47}$$

$$\left[x^-, \left[p^+, G\right]_+\right] = 2\mathrm{i}G \tag{48}$$

and all the others commutators are null. Here D, E, F, and G represent the following operators:

$$\begin{aligned} D &= x^-\,,\ p^+\,,\ x^k\,,\ p^k\ \text{ or }\ b_q^k\,,\\ E &= x^-\,,\ p^+\,,\ x^k\,,\ p^k\ \text{ or }\ \alpha_n^k\,,\\ F &= x^-\,,\ p^+\,,\ x^k\,,\ \alpha_n^k\ \text{ or }\ b_r^k\,,\\ G &= x^-\,,\ x^k\,,\ p^k\,,\ \alpha_n^k\ \text{ or }\ b_r^k\,. \end{aligned}$$

3.2 Space−time critical dimensions

3.2.1 Parabosonic case

Let us introduce, in the transverse gauge, the generators M^{i-} in the form:

$$M^{i-} = l^{i-} + E^{i-}\,, \tag{49}$$

where

$$l^{i-} = \frac{1}{2}\left[x^i, \frac{1}{p^+}\right]_+ \alpha_0^- - \frac{1}{2}\left[x^-, p^i\right]_+ \tag{50}$$

and

$$E^{i-} = -\frac{\mathrm{i}}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\left[\alpha_{-n}^i, \frac{1}{p^+}\right]_+ \alpha_n^- - \alpha_{-n}^-\left[\alpha_n^i, \frac{1}{p^+}\right]_+\right). \tag{51}$$

If we take the mean values on the physical states $\left(\alpha_{-m}^k\left|0\right\rangle\right)$ and by the use of the trilinear relations (34–38), after a lengthy and tedious calculation, the equation:

$$\left\langle 0\right|\alpha_m^l\left[M^{i-}, M^{j-}\right]\alpha_{-m}^k\left|0\right\rangle = 0 \tag{52}$$

leads to:

$$\left[M^{i-},M^{j-}\right]=-\frac{1}{2\left(p^{+}\right)^{2}}\sum_{n=1}^{\infty}\left(\left[\alpha_{-n}^{i},\alpha_{n}^{j}\right]_{+}-\left[\alpha_{-n}^{j},\alpha_{n}^{i}\right]_{+}\right)\times$$
$$\times\left(-2n+Q\frac{D-2}{12}\left(n-\frac{1}{n}\right)+2\alpha\left(0\right)\cdot\frac{1}{n}\right)=0\,. \quad (53)$$

In conclusion, to have $\left(\left[M^{i-},M^{j-}\right]=0\right)$, equation (53) gives

$$\begin{cases} D=2+24/Q\,, \\ \alpha(0)=1\,. \end{cases} \quad (54)$$

3.2.2 Paraspinning case

Let us introduce, in the transverse gauge, the generators M^{i-} in the form:

$$M^{i-}=M_{0}^{i-}\left(x\right)+K^{i-}\,, \quad (55)$$

where

$$M_{0}^{i-}\left(x\right)=l^{i-}+E^{i-}\,, \quad (56)$$
$$l^{i-}=\frac{1}{2}\left[x^{i},\frac{1}{p^{+}}\right]_{+}\alpha_{0}^{-}-\frac{1}{2}\left[x^{-},p^{i}\right]_{+}\,, \quad (57)$$
$$E^{i-}=-\frac{\mathrm{i}}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\left[\alpha_{-n}^{i},\frac{1}{p^{+}}\right]_{+}\alpha_{n}^{-}-\alpha_{-n}^{-}\left[\alpha_{n}^{i},\frac{1}{p^{+}}\right]_{+}\right), \quad (58)$$

with

$$\alpha_{-n}^{-}=-\frac{1}{4}\sum_{l=-\infty}^{+\infty}\left[\alpha_{n-l}^{i},\alpha_{l}^{i}\right]_{+}-\frac{1}{4}\sum_{r=-\infty}^{+\infty}\left(r-\frac{n}{2}\right)\left[b_{n-r}^{i},b_{r}^{i}\right]-\frac{a}{2}\delta_{n,0} \quad (59)$$

and

$$K^{i-}=-\frac{\mathrm{i}}{4}\sum_{r=\frac{1}{2}}^{\infty}\left(\left[b_{-r}^{i},\frac{1}{p^{+}}\right]_{+}G_{r}-G_{-r}\left[b_{r}^{i},\frac{1}{p^{+}}\right]_{+}\right), \quad (60)$$

with

$$G_{r}=\frac{1}{2}\sum_{n=-\infty}^{+\infty}\left[\alpha_{n}^{i},b_{r}^{i}\right]_{+}\,. \quad (61)$$

Now, in the same way, projecting the equation $\left[M^{i-},M^{j-}\right]=0$ on the physical states $\alpha_{-m}^{k}\left|0\right\rangle$ and $b_{-s}^{k}\left|0\right\rangle$ and with the solely use of the trilinear relations (42–48), the equation:

$$\left\langle 0\right|\alpha_{m}^{l}\left[M^{i-},M^{j-}\right]\alpha_{-m}^{k}\left|0\right\rangle+\left\langle 0\right|b_{s}^{l}\left[M^{i-},M^{j-}\right]b_{-s}^{k}\left|0\right\rangle=0 \quad (62)$$

leads to the result:

$$\begin{aligned}\left[M^{i-},M^{j-}\right] = & \frac{1}{2\left(p^{+}\right)^{2}}\sum_{n=1}^{\infty}\left(\left[\alpha_{-n}^{i},\alpha_{n}^{j}\right]_{+}-\left[\alpha_{-n}^{j},\alpha_{n}^{i}\right]_{+}\right)\times\\ & \times\left(Q\frac{D-2}{8}\left(n-\frac{1}{n}\right)+\frac{2a}{n}-n\right)-\\ & -\frac{1}{2\left(p^{+}\right)^{2}}\sum_{r=1/2}^{\infty}\left(\left[b_{-r}^{i},b_{r}^{j}\right]_{-}-\left[b_{-r}^{j},b_{r}^{i}\right]_{-}\right)\times\\ & \times\left(\left(Q\frac{D-2}{8}-1\right)\left(r^{2}-\frac{1}{4}\right)+2a-1\right)=0\,.\end{aligned} \tag{64}$$

We are thus led to conclude that, in order to have $\left[M^{i-},M^{j-}\right]=0$, one must have:

$$\begin{cases} D=2+8/Q\,,\\ a=1/2\,.\end{cases} \tag{65}$$

In particular, one can have paraspinning string with the critical dimensions $D=10$, 6, 4, 3 resp. in the orders $Q=1$, 2, 4, 8. This coincide with the dimensions in which the classical superstrings can be formulated. We investigate then the existence possibilities of the $D=3,4,6$ parasuperstring.

4 $D=3$, 4, 6 parasuperstring

Let us first introduce the notations we will use in this work through a brief summary of some familiar results in superstring theory.

In the transverse gauge, the action is postulated as:

$$S=-\frac{1}{2\pi}\int \mathrm{d}\sigma\,\mathrm{d}\tau\left(\partial_{a}X^{i}\partial^{a}X_{i}-i\overline{S}^{Aa}\rho^{B}\partial_{B}S^{Aa}\right), \tag{66}$$

where: A, $B=1$, 2 and a is the components index of the spinor S^{A} with

$$S=\begin{pmatrix}S^{1}\\ S^{2}\end{pmatrix},\qquad \overline{S}=S^{T}\rho^{1}$$

and

$$\rho^{1}=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\qquad \rho^{2}=\begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\qquad \left\{\rho^{A},\rho^{B}\right\}=-2\eta^{AB}\,.$$

The solutions are:

$$X^{i}\left(\sigma,\tau\right)=x^{i}+p^{i}\tau+\sum_{n\neq 0}\frac{1}{n}\alpha_{n}^{i}\left(0\right)\exp\left(-\mathrm{i}n\tau\right)\cos n\sigma\,, \tag{67}$$

$$\begin{cases} S^{1a}\left(\sigma,\tau\right)=\dfrac{1}{\sqrt{2}}\displaystyle\sum_{n\in\mathbb{Z}}S_{n}^{1a}\exp\left[-\mathrm{i}n\left(\tau-\sigma\right)\right],\\ S^{2a}\left(\sigma,\tau\right)=\dfrac{1}{\sqrt{2}}\displaystyle\sum_{n\in\mathbb{Z}}S_{n}^{2a}\exp\left[-\mathrm{i}n\left(\tau+\sigma\right)\right],\end{cases} \tag{68}$$

the two supersymmetric generators are given by:

$$Q^a = \left(2p^+\right)^{1/2} s_0^a \,, \tag{69}$$

$$Q^{\dot a} = \left(p^+\right)^{-1/2} \gamma^i_{\dot a a} \sum_{n=1}^{\infty} s^a_{-n} \alpha^i_n \,, \tag{70}$$

which verify the anticommutation relations:

$$\begin{aligned} \left[Q^a, Q^b\right]_+ &= 2p^+ \delta^{ab} \,, \\ \left[Q^a, Q^{\dot a}\right]_+ &= \sqrt{2}\gamma^i_{\dot a a} p^i \,, \\ \left[Q^{\dot a}, Q^{\dot b}\right]_+ &= 2H\delta^{\dot a \dot b} \,, \end{aligned} \tag{71}$$

where the hamiltonian H is given by:

$$H = \frac{1}{p^+} \left\{ \sum_{n=1}^{\infty} \left(\alpha^i_{-n}\alpha^i_n + n s^a_{-n} s^a_n\right) + \frac{1}{2} p_i^2 \right\} . \tag{72}$$

4.1 Parabose−parafermi superstring

4.1.1 Poincaré parasuperalgebra

In connection with some works on the extension of the supersymmetric quantum mechanic (SSQM) to the paraSSQM (see for example [4], [5], [6], ... and the references herein), our system correspond to a parabose–parafermi supersymmetric system in the sense that despite of the parastatistical algebraic structure of the dynamical variables, the combined set of the generators of the symmetries forms the algebra of the SSQM. Let us first set the formalism, in the transverse gauge, the paraquantum operators x^-, p^+, x^i, p^i, α^i_n and s^a_n verify the trilinear relations:

$$\left[s^a_n \,, \left[s^b_m \,, s^c_l \,\right]_-\right] = 2\left(\delta^{ab}\delta_{n+m} s^c_l \, - \delta^{ac}\delta_{n+l} s^b_m \,\right), \tag{73}$$

$$\left[\alpha^i_n, \left[\alpha^j_m, \alpha^k_l\right]_+\right] = 2\left(n\delta^{ij}\delta_{n+m,0}\alpha^k_l + n\delta^{ik}\delta_{n+l,0}\alpha^j_m\right), \tag{74}$$

$$\left[x^i, \left[p^j, p^k\right]_+\right] = 2\mathrm{i}\left(\delta^{ij}p^k + \delta^{ik}p^j\right), \tag{75}$$

$$\left[\alpha^i_n, \left[\alpha^j_m, A\right]_+\right] = 2\delta^{ij} n \delta_{n+m} A \,, \tag{76}$$

$$\left[s^a_n \,, \left[s^b_m \,, B\right]_+\right]_+ = 2\delta^{ab}\delta_{n+m} B \,, \tag{77}$$

$$\left[x^i, \left[p^j, C\right]_+\right] = 2\mathrm{i}\delta^{ij} C \,, \tag{78}$$

$$\left[x^-, \left[p^+, D\right]_+\right] = 2\mathrm{i} D \,, \tag{79}$$

where A, B, C, and D represent the operators:

$$\begin{aligned}A &= x^- \ , \ p^+ \ , \ x^k \ , \ p^k \ \text{ or } \ s_l^a \ , \\ B &= x^- \ , \ p^+ \ , \ x^i \ , \ p^i \ \text{ or } \ \alpha_n^i \ , \\ C &= x^- \ , \ p^+ \ , \ x^k \ , \ \alpha_n^k \ \text{ or } \ s_n^a \ , \\ D &= x^- \ , \ x^i \ , \ p^i \ , \ \alpha_n^i \ \text{ or } \ s_n^a \ ,\end{aligned}$$

all the others commutators are null.

In terms of the Green components, one can write:

$$\left[x^{i(\alpha)}, p^{j(\alpha)}\right] = \mathrm{i}\delta^{ij} \ , \qquad \left[x^{i(\alpha)}, p^{j(\beta)}\right]_+ = 0 \ , \quad \alpha \neq \beta \ , \tag{80}$$

$$\left[x^{-(\alpha)}, p^{+(\alpha)}\right] = \mathrm{i} \ , \qquad \left[x^{-(\alpha)}, p^{+(\beta)}\right]_+ = 0 \ , \quad \alpha \neq \beta \ , \tag{81}$$

$$\left[\alpha_n^{i(\alpha)}, \alpha_m^{j(\alpha)}\right] = n\delta^{ij}\delta_{n+m,0} \ , \qquad \left[\alpha_n^{i(\alpha)}, \alpha_m^{j(\beta)}\right]_+ = 0 \ , \quad \alpha \neq \beta \ , \tag{82}$$

$$\left[s_n^{a(\alpha)}, s_m^{b(\alpha)}\right]_+ = \delta^{ab}\delta_{n+m,0} \ , \qquad \left[s_n^{a(\alpha)}, s_m^{b(\beta)}\right] = 0 \ , \quad \alpha \neq \beta \ , \tag{83}$$

$$\left[\alpha_n^{i(\alpha)}, s_m^{a(\alpha)}\right] = 0 \ , \qquad \left[\alpha_n^{i(\alpha)}, s_m^{a(\beta)}\right]_+ = 0 \ , \quad \alpha \neq \beta \tag{84}$$

and all the other commutators (anticommutators) of the type $\left[A^{(\alpha)}, B^{(\alpha)}\right]$, $\left[A^{(\alpha)}, B^{(\beta)}\right]_+$, for $\alpha \neq \beta$ are null.

The first question asqued here is about the algebra verified by the supersymmetric generators. Indeed, because of the relation

$$\left[p^+, \left[s_n^a, \alpha_m^i\right]_+\right] = 0 \ , \tag{85}$$

one can set

$$Q^a = \frac{1}{2}\left[\left(2p^+\right)^{1/2}, s_0^a\right]_+ \ , \tag{86}$$

$$Q^{\dot{a}} = \left(p^+\right)^{-1/2}\gamma_{\dot{a}a}^i \sum_{n=-\infty}^{\infty} \frac{1}{2}\left[s_{-n}^a, \alpha_n^i\right]_+ \ . \tag{87}$$

And again, because of the relations

$$\begin{aligned}\left[p^+, \left[\alpha_n^i, \alpha_m^j\right]_+\right] &= 0 \ , \\ \left[p^+, \left[s_n^a, s_m^b\right]_-\right] &= 0 \ , \\ \left[p^+, \left[p_i, p_i\right]_+\right] &= 0 \ .\end{aligned} \tag{88}$$

The Hamiltonian is

$$H = \frac{1}{p^+}\left\{\frac{1}{2}\sum_{n=1}^{\infty}\left(\left[\alpha_{-n}^i, \alpha_n^i\right]_+ + n\left[s_{-n}^a, s_n^a\right]_-\right) + \frac{1}{2}p_i^2\right\} \tag{89}$$

by the use of the trilinear relations (53) one can demonstrate that

$$\left[Q^a, Q^b\right]_+ = 2p^+\delta^{ab}\,, \tag{90}$$

$$\left[Q^{\dot a}, Q^{\dot b}\right]_+ = 2\delta^{\dot a\dot b}H\,, \tag{91}$$

$$\left[Q^a, Q^{\dot a}\right]_+ = \sqrt{2}\gamma^i_{\dot a a}p^i \tag{92}$$

so that we retrouve the same relations (71) as in the supersymmetric quantum case.

4.1.2 Partition function

The paraquantum mass operator is defined as:

$$M^2 = \frac{1}{2\alpha'}\sum_{n=1}^{\infty}\left(\left[\alpha^i_{-n}, \alpha^i_n\right]_+ + n\left[s^a_{-n}\ , s^a_n\ \right]_-\right). \tag{93}$$

In order to determine the spectrum, let us first construct the partition function. To do this, we adopt the same procedure as in the ordinary case. We treat separately the parabosonic and the parafermionic cases and we postulate the parasuperstring partition function as follows:

$$\begin{aligned} f\left(x\right) &= \operatorname{Tr} x^{M^2} = \operatorname{Tr} x^{\frac{1}{2\alpha'}\sum_{n=1}^{\infty}\left(\left[\alpha^i_{-n}, \alpha^i_n\right]_+ + n\left[s^a_{-n}, s^a_n\right]_-\right)} = \\ &= \sum_{n'=0}^{\infty}\left\langle n'\right| x^{\sum_{n=1}^{\infty} n\left(\left[a^{i+}_n, a^i_n\right]_+ + \left[s^a_{-n}, s^a_n\right]_-\right)}\left|n'\right\rangle, \end{aligned} \tag{94}$$

with $\alpha_n = \sqrt{n}a_n$, $\alpha_{-n} = \sqrt{n}a^+_n$ and $2\alpha' = 1$.

One can then calculate this expression and find:

$$f(x) = 2(D-2)\prod_{n=1}^{\infty}\left(\frac{1 + x^{2n} + x^{4n} + \cdots + x^{2nQ}}{1 - x^{2n}}\right)^{D-2}\,, \tag{95}$$

so that the explicit forms in the cases

$$\begin{aligned} D = 3 &\rightarrow Q = 8\,, \\ D = 4 &\rightarrow Q = 4\,, \\ D = 6 &\rightarrow Q = 2\,, \\ D = 10 &\rightarrow Q = 1 \end{aligned}$$

are given in table 1.

Now, one can describe the spectrum through the fundamental and the three firsts levels and verify the consistency of the developments in the relations (Table 1). By the use of both the trilinear relations and the Green decomposition, we calculate the masses of the first levels of the physical states and verify that the mass did not depend on the space–time dimensions, in the same time, this suggests a general form of these physical states. Let us determine the number of states of each level considered.

Table 1. $D = 3$, 4, 6, 10 partition functions

D	$f(x)$
$D=3$	$2\left[1+2x^2+5x^4+10x^6+20x^8+30x^{10}+56x^{12}+O(x^{14})\right]$
$D=4$	$4\left[1+4x^2+14x^4+40x^6+105x^8+196x^{10}+486x^{12}+O(x^{14})\right]$
$D=6$	$8\left[1+8x^2+44x^4+188x^6+694x^8+1640x^{10}+5688x^{12}+O(x^{14})\right]$
$D=10$	$16[1+16x^2+144x^4+960x^6+5264x^8+25\,056\,x^{10}+106\,944\,x^{12}+O(x^{14})]$

4.1.3 Spectrum

Fundamental state

One sets $|0\rangle$ for $|i\rangle$ or $|a\rangle$, where

$$\begin{array}{ll} |i\rangle & \text{is the } (D-2) \text{ physical transverse polarizations of the massless vector field} \\ |a\rangle & \text{is the } (D-2) \text{ components spinor partner} \end{array} \tag{96}$$

the total number is $2(D-2)$ states.

first level

In the same way, the first level states are:

$$\begin{array}{lll} s^a_{-1}\,|i\rangle & (D-2)^2 & \text{parafermion states} \\ s^a_{-1}\,|b\rangle & (D-2)^2 & \text{paraboson states} \\ \alpha^i_{-1}\,|j\rangle & (D-2)^2 & \text{paraboson states} \\ \alpha^i_{-1}\,|a\rangle & (D-2)^2 & \text{parafermion states} \end{array} \tag{97}$$

The total paraboson and parafermion states is $4(D-2)^2$.

2^{nd} level

The 2^{nd} level states and their numbers are determined and listed in table 2:

Table 2. 2^{nd} level states and their numbers

$\left[s^a_{-1}\,,s^b_{-1}\,\right]_+\|0\rangle$	$\left[(D-2)+\frac{(D-2)(D-3)}{2}\right][2(D-2)]$
$\left[\alpha^i_{-1},\alpha^j_{-1}\right]_+\|0\rangle$	$\left[(D-2)+\frac{(D-2)(D-3)}{2}\right][2(D-2)]$
$\left[\alpha^i_{-1},s^a_{-1}\right]_+\|0\rangle$	$(D-2)^2\,[2(D-2)]$
$\alpha^i_{-2}\,\|0\rangle$	$(D-2)\,[2(D-2)]$
$s^a_{-2}\,\|0\rangle$	$(D-2)\,[2(D-2)]$

Then, the total number is:

$$\left[(D-2)^2+4(D-2)+(D-2)(D-3)\right][2(D-2)]\ . \tag{98}$$

3^{rd} level

The states are defined as in the table 3.

Table 3. 3^{rd} level states

$\frac{1}{2}\alpha^i_{-1}\left[\alpha^j_{-1}, s^a_{-1}\right]_+ \lvert 0\rangle \leftrightarrow \frac{1}{2} s^a_{-1}\left[\alpha^i_{-1}, \alpha^j_{-1}\right]_+ \lvert 0\rangle$	(a)
$\frac{1}{2} s^a_{-1}\left[\alpha^i_{-1}, s^b_{-1}\right]_+ \lvert 0\rangle \leftrightarrow \frac{1}{2}\alpha^i_{-1}\left[s^a_{-1}\ , s^b_{-1}\ \right]_+ \lvert 0\rangle$	(b)
$\frac{1}{2} s^a_{-1}\left[s^b_{-1}\ , s^c_{-1}\ \right]_+ \lvert 0\rangle$	(c)
$\frac{1}{2}\alpha^i_{-1}\left[\alpha^j_{-1}, \alpha^k_{-1}\right]_+ \lvert 0\rangle$	(d)
$\frac{1}{2}\left[\alpha^i_{-1}, s^a_{-2}\right]_+ \lvert 0\rangle \quad , \quad \frac{1}{2}\left[\alpha^i_{-2}, s^a_{-1}\right]_+ \lvert 0\rangle$	(e)
$\frac{1}{2}\left[\alpha^i_{-2}, \alpha^j_{-1}\right]_+ \lvert 0\rangle \quad , \quad \frac{1}{2}\left[s^a_{-2}\ , s^b_{-1}\ \right]_+ \lvert 0\rangle$	(f)
$s^a_{-3}\lvert 0\rangle \qquad , \qquad \alpha^i_{-3}\lvert 0\rangle$	(g)

Notice here that, in order to determine the number of states, we take into account the equivalence between the left and the right hand sides of the relations (a), (b) and again the fact that, in the order $Q = 2$ of the paraquantization, $(s^a_{-1})^3 = 0$ (in the relation (c)). One can then determine the numbers of states for each case and obtain the results listed in the table 4.

Table 4. The number of each 3^{rd} level state

(a)	$(D-2\left[(D-2)+\frac{(D-2)(D-3)}{2}\right][2(D-2)]$
(b)	$(D-2\left[(D-2)+\frac{(D-2)(D-3)}{2}\right][2(D-2)]$
(c)	$\nearrow \left[(D-2)+(D-2)(D-3)+\frac{(D-2)(D-3)(D-4)}{3!}\right][2(D-2)] \qquad D\neq 6$ $\searrow \left[(D-2)(D-3)+\frac{(D-2)(D-3)(D-4)}{3!}\right][2(D-2)]=128 \qquad with \quad D=6 \quad Q=2\Rightarrow(s^a)^3=0$
(d)	$\left[(D-2)+(D-2)(D-3)+\frac{(D-2)(D-3)(D-4)}{3!}\right][2(D-2)]$
(e)	$2\left[(D-2)^2 2(D-2)\right][2(D-2)]$
(f)	$2\left[(D-2)^2 2(D-2)\right][2(D-2)]$
(g)	$2(D-2)\,[2(D-2)]$

The total number of the paraboson and the parafermion states is deduced.

Finally, one can recapitulate all the results and list them in the table 5.

We find a common chord between the number of states and the one given by the partition functions.

4.2 Bose−parafermi case

In the last point and in connection with some works on the fractional superstrings (see for example [7], [8] and the references herein) based on a bosonic–fractional fermionic system which are again formulated in $D = 2,\ 3,\ 4,\ 6,\ 10$, a boson–parafermi superstrings are considered. Unlike the first case, the combined set of the generators of the symmetries forms the algebra of the PSSQM in the sense of Beckers and Debergh. Indeed, in the transverse gauge, the operators x^-, p^+, x^i,

Table 5. The total numbers of the first levels states in concordance with the partition function developments in $D = 3,\ 4,\ 6$

	fund.state	1^{st} level	2^{nd}	3^{rd}	partition function
$D = 3$	2	4	10	20	$2 + 4x^2 + 10x^4 + 20x^6 +$
$D = 4$	4	16	56	160	$4 + 16x^2 + 56x^4 + 160x^6 +$
$D = 6$	8	64	342	1504	$8 + 64x^2 + 342x^4 + 1504x^6 +$

p^i, α^i_n and s^a_n verify the following commutation relations:

$$\left[s^a_n, \left[s^b_m, s^c_l\right]_-\right] = 2\left(\delta^{ab}\delta_{n+m}s^c_l - \delta^{ac}\delta_{n+l}s^b_m\right), \tag{99}$$

$$\left[\alpha^i_n, \alpha^j_m\right] = n\delta^{ij}\delta_{n+m,0}\,, \tag{100}$$

$$\left[x^i, p^j\right] = \mathrm{i}\delta^{ij}\,, \tag{101}$$

$$\left[x^-, p^+\right] = \mathrm{i} \tag{102}$$

and all the other commutators are null. Let us determine the algebra verified by the supersymmetric generators:

One sets

$$Q^a = \left(2p^+\right)^{1/2} s^a_0\,, \tag{103}$$

$$Q^{\dot{a}} = \left(p^+\right)^{-1/2}\gamma^i_{\dot{a}a}\sum_{n=-\infty}^{\infty} s^a_{-n}\alpha^i_n \tag{104}$$

and

$$H = \frac{1}{p^+}\left\{\sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\,n\left[s^a_{-n}, s^a_n\right]_-\right) + \frac{1}{2}\,p^2_i\right\}. \tag{105}$$

Now, by the only use of the relations(99–102) one can prove that:

$$\left[Q^a\ , \left[Q^b\ , Q^c\right]_-\right] = 2\left(2p^+\right)\left[\delta^{ab}Q^c\ - \delta^{ac}Q^b\right], \tag{106}$$

$$\left[Q^a\ , \left[Q^b\ , Q^{\dot{a}}\right]_-\right] = 2p^+\delta^{ab}Q^{\dot{a}} - \sqrt{2}\gamma^i_{\dot{a}a}p^iQ^b\,, \tag{107}$$

$$\left[Q^{\dot{a}}\ , \left[Q^a\ , Q^b\right]_-\right] = \sqrt{2}\gamma^i_{\dot{a}a}p^iQ^b - \sqrt{2}\gamma^j_{\dot{a}b}p^jQ^a\,, \tag{108}$$

$$\left[Q^{\dot{a}}\ , \left[Q^{\dot{b}}\ , Q^a\right]_-\right] = 2\delta^{\dot{a}\dot{b}}HQ^a - \sqrt{2}\gamma^i_{\dot{a}a}p^iQ^{\dot{b}}\,, \tag{109}$$

$$\left[Q^a\ , \left[Q^{\dot{a}}\ , Q^{\dot{b}}\right]_-\right] = \sqrt{2}\gamma^i_{\dot{a}a}p^iQ^{\dot{b}} - \sqrt{2}\gamma^j_{\dot{a}b}p^jQ^{\dot{a}}\,. \tag{110}$$

This correspond to the algebra of the parasupersymmetric quantum mechanic.

5 Conclusion

The first result obtained here is that the general form suggested to the physical states is coherent with the $D = 3$, 4, 6 partition functions. The second result is in concordance with what we find in the literature. Indeed, two types of systems are considered, a parabose–parafermi supersymmetric system and a bose–parafermi supersymmetric system. The relations (90–92) imply that the first one corresponds to an ordinary supersymmetric system while the relations (106–110) imply that the second one corresponds to a parasupersymmetric system. Of course, one can verify here that the set of relations (90–92) imply those of (106–110), but the inverse is, in general, not true. This is completely consistent with the fact that bosons are particular parabosons.

Acknowledgments. One of the authors (N.B) would like to thank the organizers of this conference, particularly prof. Cestmir Burdík.

This work was supported by the Algerian Ministry of Education and Research under contracts No.D2501/14/2000 and No.D1602/04/2004.

References

[1] F. Ardalan and F. Mansouri: Phys. Rev. D **9** (1974) 3341.

[2] N. Belaloui and H. Bennacer: Czech. J. Phys. **53** (2003) 769.

[3] N. Belaloui and H. Bennacer: Czech. J. Phys. **54** (2004) 621.

[4] J. Beckers and N.Debergh: J. Math. Phys. **31** (1990) 1513.

[5] Z. Symenoh: Nonlinear Math. Phys. **3** (1996) 181.

[6] A. Mostafazadeh: Int. J. Mod. Phys. A **11** (1996) 2941.

[7] Keith R. Dienes: Nucl. Phys. B **413** (1994) 103.

[8] Philip C. Argyres and Keith R. Dienes: Phys. Rev. Lett. **71** (1993) 819.