Algebraic construction of integrable and super integrable hierarchies

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A general construction of integrable hierarchies based on affine Lie algebras is presented. The models are specified according to some algebraic data and their time evolution is obtained from solutions of the zero curvature condition. Such framework provides an unified treatment of relativistic and non relativistic models. The extension to the construction of supersymmetric integrable hierarchies is proposed. An explicit example of $N = 2$ super mKdV and sinh–Gordon is presented.

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1 Introduction

Integrable models consist of a very peculiar class of physical systems described by non linear differential equations. In two dimensional space time they present an infinite number of conservation laws and admit soliton solutions. Well known examples such the sine–Gordon or mKdV equations have been shown to be connected to many applications in several branches of physics.

Integrable hierarchies can be constructed and classified in terms of an affine Lie algebraic structure. Equations of motion for relativistic integrable models can be derived from the Leznov–Saveliev's approach [1], which was latter discovered to be connected to reductions of the Wess–Zumino–Witten model [2]. For non relativistic integrable hierarchies, the Lax formulation proved to be an important framework which also relies in the algebraic decomposition of an affine Lie algebra.

The aim of this paper is to discuss a universal formulation were both relativistic and non relativistic integrable models can be viewed. The main ingredient is the zero curvature condition which employs, in a natural manner, Lie algebra valued functionals of the physical fields.

We first discuss the Leznov–Saveliev construction of relativistic integrable models in terms of a decomposition of an affine Lie algebra. The models are shown to be classified according to some algebraic data, namely $(Q, \epsilon_\pm, \mathcal{G}_0^0)$, where Q specifies the space of physical fields, $\epsilon_\pm$, the interaction and $\mathcal{G}_0^0 = \mathcal{K} = \mathrm{Ker}(\mathrm{ad}_{\epsilon_+})$ containing the internal soliton symmetries. Next, we discuss the Lax formulation for the non-relativistic integrable models in terms of certain positive grade elements in $\mathcal{K}$, each of them is assigned to a time evolution. The precise correspondence between those two formulations is explained in Sect. 3 where we introduce negative grade time evolution and show that the relativistic formulation of Leznov and Saveliev corresponds to the grade -1 generator in $\mathcal{K}$. In Sect. 4 we extend both formulations to accommodate affine decompositions with integer and half integer gradings. This leads to the construction of supersymmetric integrable hierarchies. Finally as an example, we construct the $N=2$ super MKdV and $N=2$ super sinh–Gordon models associated to $\widehat{SL}(2,2)$ showing that they indeed belong to the same integrable hierarchy.

2 General construction of relativistic integrable hierarchies

A general construction of relativistic integrable hierarchies in terms of an affine Lie algebra $\hat{\mathcal{G}}$ can be established from the Leznov–Saveliev equations of motion [1],

$$\bar{\partial}(B^{-1}\partial B) + [\epsilon_-, B^{-1}\epsilon_+ B] = 0\,, \quad \partial(\bar{\partial}BB^{-1}) - [\epsilon_+, B\epsilon_- B^{-1}] = 0\,, \tag{1}$$

where the space time is represented by of the light cone coordinate $z = t + x$, $\bar{z} = t - x$ and a decomposition of $\hat{\mathcal{G}}$ into graded subspaces $\hat{\mathcal{G}} = \oplus_a \mathcal{G}_a$, $a \in Z$ according to a grading operator Q such that $[Q, \mathcal{G}_a] = a\mathcal{G}_a$ is assumed. Here B represents an element of the zero grade subgroup $B \in G_0$ and is parametrized by the physical fields (Toda fields) of the theory. $\epsilon_\pm$ are constant generators of grade ± 1 which characterize the non linear interaction. The classical integrability of these models follows from their zero curvature (Lax) representation:

$$\partial\bar{\mathcal{A}} - \bar{\partial}\mathcal{A} + [\mathcal{A}, \bar{\mathcal{A}}] = 0\,, \quad \mathcal{A}, \bar{\mathcal{A}} \in \oplus_{i=0,\pm 1}\mathcal{G}_i\,, \tag{2}$$

with

$$\mathcal{A} = B\epsilon_- B^{-1}\,, \quad \bar{\mathcal{A}} = -\epsilon_+ - \bar{\partial}BB^{-1}\,. \tag{3}$$

The existence of an infinite set (of commuting) conserved charges P_m, $m = 0, 1, \cdots$ is a direct consequence of eqn. (2), namely,

$$P_m(t) = \mathrm{Tr}\big(T(t)\big)^m\,, \quad \partial_t P_m = 0\,, \quad T(t) = \lim_{L\to\infty} \mathcal{P}\exp\left(\int_{-L}^{L} \mathcal{A}_x(t,x)\mathrm{d}x\right),$$

where $\mathcal{A}_x = \mathcal{A} + \bar{\mathcal{A}}$. Well known examples of integrable models fall into the general construction as we shall now detail.

Consider $\hat{\mathcal{G}} = \widehat{SL}(2)$ with the principal gradation, i.e. $Q = 2\mathrm{d} + \frac{1}{2}H$ and d is the derivation operator. The zero grade subalgebra is then $\mathcal{G}_0 = \{H^{(0)}\}$ and hence $B = \mathrm{e}^{\phi H^{(0)}}$. If we now choose $\epsilon_+ = E_\alpha^{(0)} + E_{-\alpha}^{(1)}$, $\epsilon_- = \epsilon_+^\dagger$ we find from (2) the sinh–Gordon equation

$$\partial\bar{\mathcal{A}} - \bar{\partial}\mathcal{A} + [\mathcal{A}, \bar{\mathcal{A}}] = \left(\bar{\partial}\partial\phi + \mathrm{e}^{2\phi} - \mathrm{e}^{-2\phi}\right) H = 0\,. \tag{4}$$

Another decomposition of $\widehat{SL}(2)$ can be obtained from the homogeneous gradation where $Q = \mathrm{d}$. The zero grade subalgebra acquires a nonabelian structure, i.e. $\mathcal{G}_0 =$

$\{H^{(0)}, E^{(0)}_{\pm\alpha}\}$ and $B = e^{RH^{(0)}/2} e^{\chi E^{(0)}_{-\alpha}} e^{\psi E^{(0)}_{\alpha}} e^{RH^{(0)}/2}$. If we now choose $\epsilon_\pm = H^{(\pm)}$ we find from eqns. (1) the existence of two chiral currents associated to the subalgebra $\mathcal{G}_0^0 \subset \mathcal{G}_0$ namely, $\bar\partial J_H = \bar\partial \,\mathrm{Tr}(H^{(0)} B^{-1}\partial B) = 0$ and $\partial \bar J_H = \partial\,\mathrm{Tr}(H^{(0)}\bar\partial B B^{-1}) = 0$, since $\mathcal{G}_0^0 = U(1) = \{H^{(0)}\}$, $[\mathcal{G}_0^0, \epsilon_\pm] = 0$. This fact allows the implementation of the following subsidiary constraints

$$J_H = \partial R - \frac{\psi\partial\chi}{\Delta} = 0\,, \quad \bar J_H = \bar\partial R - \frac{\chi\bar\partial\psi}{\Delta} = 0\,, \quad \Delta = 1 + \psi\chi\,, \tag{5}$$

which eliminates the nonlocal field R. The equations of motion for this case are then given by the zero curvature condition (2) or, equivalently by eqns. (1), when the subsidiary constraints (5) are taken into consideration (i.e. for the coset $G_0/G_0^0 = SL(2)/U(1)$), yielding the equations of motion of Lund–Regge (complex sine–Gordon) model,

$$\bar\partial\left(\frac{\partial\chi}{\Delta}\right) + \frac{\chi\partial\chi\bar\partial\psi}{\Delta^2} + 2\chi = 0\,, \quad \partial\left(\frac{\bar\partial\psi}{\Delta}\right) + \frac{\psi\partial\chi\bar\partial\psi}{\Delta^2} + 2\psi = 0\,. \tag{6}$$

The gauged $\mathcal{G}_0^0 = U(1)$ factor arisen from the constraints (5) is responsible for the global $U(1)$ symmetry $\psi \to \psi e^{\alpha}$, $\chi \to \chi e^{-\alpha}$ and henceforth for the conservation of the electric charge

$$Q_{\mathrm{el}} = \int_{-\infty}^{+\infty} (\partial_x R)\,\mathrm{d}x\,. \tag{7}$$

The above examples can be extended to higher rank affine algebras with gradations which are intermediate between the principal and homogeneous ones. Let us consider $\mathcal{G} = \widehat{SL}(n+1)$. In general, we can construct integrable hierarchies according to the following algebraic structures:

1. $\mathcal{G}_0^0 = \emptyset$ characterizes the choices of

$$Q = (N+1)\mathrm{d} + \sum_{l=1}^{N} \lambda_l \cdot H\,, \quad \mathcal{G}_0 = U(1)^N = \{h_1, \cdots, h_N\}\,,$$
$$\epsilon_\pm = \sum_{l=1}^{N} E^{(0)}_{\pm\alpha_l} + E^{(\pm 1)}_{\mp(\alpha_1+\cdots+\alpha_N)}\,,$$

which gives rise to the well known abelian affine Toda model (see for instance [2], [1]).

2. (a) $\mathcal{G}_0^0 = U(1) = \{\lambda_1 \cdot H\}$

$$Q = N\mathrm{d} + \sum_{l=2}^{N} \lambda_l \cdot H\,, \quad \mathcal{G}_0 = SL(2)\otimes U(1)^{N-1} = \{E_{\pm\alpha_1}, h_1, \cdots, h_N\}\,,$$
$$\epsilon_\pm = \sum_{l=2}^{N} E^{(0)}_{\pm\alpha_l} + E^{(\pm 1)}_{\mp(\alpha_2+\cdots+\alpha_N)}$$

corresponds to the simplest non abelian affine Toda model of dyonic type, admitting electrically charged topological solitons (see for instance [3]).

(b) $\mathcal{G}_0^0 = U(1) \otimes U(1) = \{\lambda_1 \cdot H, \lambda_N \cdot H\}$

$$Q = (n-1)\mathrm{d} + \sum_{l=2}^{N-1} \lambda_l \cdot H\,, \quad \epsilon_\pm = \sum_{l=2}^{N-1} E^{(0)}_{\pm\alpha_l} + E^{(\pm 1)}_{\mp(\alpha_2+\cdots+\alpha_{N-1})}\,,$$
$$\mathcal{G}_0 = SL(2) \otimes SL(2) \otimes U(1)^{N-2} = \{E_{\pm\alpha_1}, E_{\pm\alpha_N}, h_1, \cdots, h_N\}\,,$$

is of the same class of $U(1)^{\otimes k}$ dyonic type IM's, but now yielding multicharged solitons ([4]).

3. $\mathcal{G}_0^0 = SL(2) \otimes U(1) = \{E_{\pm\alpha_1}, \lambda_1 \cdot H, \lambda_2 \cdot H\}$

$$Q = (N-1)\mathrm{d} + \sum_{l=3}^{N} \lambda_l \cdot H\,, \quad \epsilon_\pm = \sum_{l=3}^{N} E^{(0)}_{\pm\alpha_l} + E^{(\pm 1)}_{\mp(\alpha_3+\cdots+\alpha_N)}\,,$$
$$\mathcal{G}_0 = SL(3) \otimes U(1)^{N-2} = \{E_{\pm\alpha_1}, E_{\pm\alpha_2}, E_{\pm(\alpha_1+\alpha_2)}, h_1, \cdots, h_N\}\,,$$

leads to dyonic models with non abelian global symmetries (see [5]).

In fact the integrable hierarchies are classified in terms of the gradation Q, the constant operators $\epsilon_\pm$ and by the global symmetry group described by the subalgebra $\mathcal{G}_0^0 = Ker(ad_E) = \mathcal{K}$.

3 Non relativistic construction of integrable hierarchies

We will follow the approach given in ref. [6] which associates to every *positive grade* n element $E^{(n)} \in C(\mathcal{K})$ ($C(\mathcal{K}) = \{x, y, \in \mathcal{K}, [x, y] = 0\}$) a time evolution t_n

$$\partial_{t_n}\Theta(t) = (\Theta E^{(n)} \Theta^{-1})_- \Theta(t)\,, \tag{8}$$

for the dressing matrix $\Theta = \exp\left(\sum_{i<0} \theta^{(i)}\right)$ being an exponential in $\mathcal{G}_<$ and $(\quad)_-$ represents the projection on strictly negative grades. By construction such flows commute, i.e. $[\partial_{t_m}, \partial_{t_n}]\Theta(t) = 0$. In particular for $n = 1$, $\partial_{t_1} \equiv \partial_x$, $\epsilon_+ = E$ we find

$$\begin{aligned}\partial_x(\Theta) &= (\Theta E \Theta^{-1})_- \Theta = \left[\Theta E \Theta^{-1} - (\Theta E \Theta^{-1})_+\right]\Theta = \\ &= \Theta E - \left(E + \left[\theta^{(-1)}, E\right]\right)\Theta = \\ &= \Theta E - (E + A_0)\Theta\,,\end{aligned} \tag{9}$$

where $(\quad)_+$ represents the projection on positive and zero grades and $A_0 = \left[\theta^{(-1)}, E\right]$. Clearly A_0 is in $\mathcal{M}$ (the Image of the adjoint operation $\mathrm{ad}\,(E)X = [E, X]$) and has grade zero. This leads to the dressing expression

$$\Theta^{-1}(\partial_x + E + A_0)\Theta = \partial_x + E \tag{10}$$

for the Lax operator $L = \partial_x + E + A_0$. Similarly, for higher flows we obtain

$$\Theta^{-1}\Big(\partial_{t_n} + E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)}\Big)\Theta = \partial_{t_n} + E^{(n)}\,, \tag{11}$$

where

$$(\Theta E^{(n)}\Theta^{-1})_+ = E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)}\,.$$

These dressing relations give rise to the zero–curvature conditions

$$\Big[\partial_x + E + A_0, \partial_{t_n} + E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)}\Big] = \Theta\,\Big[\partial_x + E, \partial_{t_n} + E^{(n)}\Big]\,\Theta^{-1} = 0\,, \tag{12}$$

where $D_n^{(j)} = D_{n\mathcal{K}}^{(j)} + D_{n\mathcal{M}}^{(j)} \in \mathcal{G}_j$. We therefore find

$$\begin{aligned}
&[E, D_n^{(n-1)}] + [A_0, E^{(n)}] = 0\,,\\
&[E, D_n^{(n-2)}] + [A_0, D_n^{(n-1)}] + \partial_x D_n^{(n-1)} = 0\,,\\
&\quad\vdots\\
&[A_0, D_n^{(0)}] - \partial_{t_n} A_0 + \partial_x D_n^{(0)} = 0\,.
\end{aligned} \tag{13}$$

Each equation can be decomposed into $\mathcal{K}$ and $\mathcal{M}$ components. It is clear that a local solution for $D_n^{(i)}$, $i = 0, \cdots, n$ can be found recursively starting from the highest grade eqn. in (13) until we reach the last. In particular the eqn. corresponding to the zero grade component also gives rise to the time evolution of the physical fields.

Let us reconsider the examples given in the previous section in connection with $\widehat{\mathcal{G}} = \widehat{SL}(2)$. With Q given in the principal gradation, $Q = 2\mathrm{d} + \frac{1}{2}H$ and $\epsilon_+ = E_\alpha^{(0)} + E_{-\alpha}^{(1)}$ we parametrize $A_0 = uH^{(0)} \in \mathcal{M}$ and solve eqn. (2) for $t = t_3$. After solving for $D_3^{(3)}$, $D_3^{(2)}$, $D_3^{(1)}$ and $D_3^{(0)}$ we obtain the equation of motion for the *mKdV* model,

$$\partial_{t_3} u = u_{xxx} + 6u^2 u_x\,. \tag{14}$$

The decomposition of $\widehat{\mathcal{G}} = \widehat{SL}(2)$ according to the homogeneous gradation $Q = \mathrm{d}$ leads to $A_0 = qE_a^{(0)} + rE_{-a}^{(0)} \in \mathcal{M}$ and eqn. (2) yields for $t = t_2$ the *nonlinear Schroedinger* equation (NLS),

$$\begin{aligned}
&\partial_{t_2} q + q_{xx} - 2rq^2 = 0\,,\\
&\partial_{t_2} r + r_{xx} + 2qr^2 = 0\,.
\end{aligned} \tag{15}$$

Notice that, the sinh–Gordon and the mKdV equations as well as, the Lund–Regge and the NLS equations, are constructed from the same algebraic structure, i.e., same $\widehat{\mathcal{G}} = \widehat{SL}(2)$, Q and same choice of $E = \epsilon_+$. This indicates that they belong to the same integrable hierarchy. In fact we shall prove such statement in a more precise manner, but before that, let us discuss the positive hierarchies associated to the $\widehat{\mathcal{G}} = \widehat{SL}(n+1)$ generalized models of the previous section. To each algebraic structure 1. to 3. we associate the following Lax operators:

1. $\mathcal{G}_0^0 = \mathcal{K} = \emptyset$

$$A_0 = \sum_{i=1}^{n} u_i h_i^{(0)} \,,$$

2. (a) $\mathcal{K} = U(1) = \{\lambda_1 \cdot H^{(0)}\}$

$$A_0 = q E_{\alpha_1}^{(0)} + r E_{-\alpha_1}^{(0)} + \sum_{i=2}^{n} u_i h_i^{(0)} \,,$$

   (b) $\mathcal{K} = U(1) \otimes U(1) = \{\lambda_1 \cdot H^{(0)}, \lambda_n \cdot H^{(0)}\}$

$$A_0 = q_1 E_{\alpha_1}^{(0)} + q_n E_{\alpha_n}^{(0)} + r_1 E_{-\alpha_1}^{(0)} + r_n E_{-\alpha_n}^{(0)} + \sum_{i=2}^{n-1} u_i h_i^{(0)} \,,$$

3. $\mathcal{K} = SL(2) \otimes U(1) = \{E_{\pm\alpha_1}, \lambda_1 \cdot H, \lambda_2 \cdot H\}$

$$A_0 = q_1 E_{\alpha_1+\alpha_2}^{(0)} + q_2 E_{\alpha_2}^{(0)} + r_1 E_{-\alpha_1-\alpha_2}^{(0)} + r_2 E_{-\alpha_2}^{(0)} + \sum_{i=3}^{n} u_i h_i^{(0)} \,, \tag{16}$$

they give rise to the generalized mKdV, multicomponent AKNS and constrained KP (cKP) hierarchies [7], [8], [9] respectively.

4 Negative grade time evolution

The time evolution associated to *negative grade* elements in $C(\mathcal{K})$ can be incorporated within the general construction of integrable hierarchies following the Riemann–Hilbert problem and its connection with the dressing formulation [10]. As in eqn. (8) to each element $E^{(-n)} \in \mathcal{K}$ we define the associated time evolution by

$$\frac{\partial \Theta(t)}{\partial t_{-n}} = -(BME^{(-n)}M^{-1}B^{-1})_{-}\Theta(t) \,, \tag{17}$$

where $M = \exp\left(\sum_{i>0} m^{(i)}\right)$ and $B \in G_0$, i.e., an element of the zero grade subgroup. It therefore follows that

$$\Theta \frac{\partial \Theta^{-1}}{\partial t_{-n}} = \frac{\partial}{\partial_{t_{-n}}} + (BME^{(-n)}M^{-1}B^{-1})_{-} \,. \tag{18}$$

By construction $[\partial_{t_{-n}}, \partial_{t_m}] = 0$ and henceforth

$$\left[\partial_x + E + A_0, \partial_{t_{-n}} + \sum_{i=1}^{n} D^{(-i)}\right] = \Theta\left[\partial_x + E, \partial t_{-n}\right]\Theta^{-1} = 0 \,. \tag{19}$$

Decomposing the zero curvature condition (19) into graded components we find

$$\begin{aligned}
&\partial_x D_n^{(-n)} + [A_0, D_n^{(-n)}] = 0\,,\\
&\partial_x D_n^{(-n+1)}[E, D_n^{(-n)}] + [A_0, D_n^{(-n+1)}] = 0\,,\\
&\quad\vdots\\
&\partial_x D_n^{(-1)}[E, D_n^{(-2)}] + [A_0, D_n^{(-1)}] = 0\,,\\
&\partial_{t_{-n}} A_0 - [E, D_n^{(-1)}] = 0\,.
\end{aligned} \tag{20}$$

Eqns. (20) can be solved recursively, however notice that in general, contrary to $D^{(i)}$ in eqns. (13), the $D^{(-i)}$ are nonlocal functionals of the fields A_0. There is one particular case, for $t = t_{-1}$, in which we obtain a closed local solution. Let $n = 1$ in eqn. (19)

$$\Big[\partial_x + E + A_0, \partial_{t_{-1}} + (BE^{(-1)}B^{-1})\Big] = 0\,. \tag{21}$$

If we now compare eqn. (21) with (2) and (3) identifying $\bar{z} = -x$, $z = t_{-1}$ and $E^{(\pm 1)} = \epsilon_\pm$ we find,

$$D_1^{(-1)} = B\epsilon_- B^{-1}\,, \quad A_0 = \bar{\partial} B B^{-1} = -\partial_x B B^{-1}\,. \tag{22}$$

With the space time identified as above, it becomes clear that the Leznov–Saveliev eqns. (1) can be put within the general construction for integrable hierarchies associated to negative grade time evolution (19). This explains the relationship between the sinh–Gordon (4) and the mKdV (14) equations as well as, the Lund–Regge (6) and the AKNS (15) equations.

5 Supersymmetric integrable hierarchies

In this section we shall consider how the structure of the Lax operators changes when terms with half–integer grades appear in $\widehat{\mathcal{G}} = \oplus_{n\in Z}\,\mathcal{G}_{n/2}$. As a consequence of such terms being present in the exponent of the dressing matrices

$$\begin{aligned}
\Theta &= \exp\Big(\sum_{i<0}\theta^{(i)}\Big) = \exp\Big(\theta^{(-1/2)} + \theta^{(-1)} + \theta^{(-3/2)} + \dots\Big)\,,\\
M &= \exp\Big(\sum_{i>0} m^{(i)}\Big) = \exp\Big(m^{(1/2)} + m^{(1)} + m^{(3/2)} + \dots\Big)
\end{aligned} \tag{23}$$

the form of the Lax operator is changed as follows (compare with (9)):

$$\begin{aligned}
\partial_{t_1}(\Theta) &= (\Theta E\Theta^{-1})_-\Theta = \big[\Theta E\Theta^{-1} - (\Theta E\Theta^{-1})_+\big]\Theta =\\
&= \Theta E + (E + \Big[\theta^{(-1)}, E\Big] + \Big[\theta^{(-1/2)}, E\Big] + \frac{1}{2}\Big[\theta^{(-1/2)}, \Big[\theta^{(-1/2)}, E\Big]\Big])\Theta =\\
&= \Theta E + (E + A_0 + A_{1/2} + k_0)\Theta\,.
\end{aligned} \tag{24}$$

Here

$$A_0 = \Big[\theta^{(-1)}, E\Big] + \frac{1}{2}\Big[\theta^{(-1/2)}, \Big[\theta^{(-1/2)}, E\Big]\Big]\Big|_{\mathcal{M}} \in \mathcal{M}\,, \tag{25}$$

$$A_{1/2} = \Big[\theta^{(-1/2)}, E\Big] \in \mathcal{M}\,, \tag{26}$$

$$k_0 = \frac{1}{2}\Big[\theta^{(-1/2)}, \Big[\theta^{(-1/2)}, E\Big]\Big]\Big|_{\mathcal{K}} \in \mathcal{K}\,, \tag{27}$$

where $\Big|_{\mathcal{K}}$ and $\Big|_{\mathcal{M}}$ denote projections on the kernel $\mathcal{K}$ and image $\mathcal{M}$, respectively. This shows that, in case of a half-integer grading, a general expression for the Lax operator is

$$\mathcal{L} = \partial_x + E + A_0 + A_{1/2} + k_0\,. \tag{28}$$

The unconventional grade zero term k_0 residing in $\mathcal{K}$ appears here due to the half–integer grading (encountered in case of $sl(2|1)$ with principal gradation [11]).

Following the procedure explained in Sect. 2 with Θ given in (23) we generalize the zero curvature form of eqn. (12),i.e.

$$\begin{aligned}\Big[\partial_x + E + A_0 + A_{1/2} + k_0, \partial_{t_n} + E^{(n)} + \sum_{i=0}^{2n-1} D_n^{(i/2)}\Big] &= \\ = \Theta\left[\partial_x + E, \partial_{t_n} + E^{(n)}\right]\Theta^{-1} &= 0\,,\end{aligned} \tag{29}$$

which can be solved recursively for $D_n^{(i)}$ and $D_n^{(i+1/2)}$, $i = 1, \cdots, n-1$. This also leads to the equations of motion (time evolution) for the fields A_0.

The negative grade sector (19) can also be extended. From Sect. 3 we also find

$$\begin{aligned}\Big[\partial_x + E + A_0 + A_{1/2} + k_0, \partial_{t_{-n}} + \sum_{i=1}^{2n} D_n^{(-i/2)}\Big] &= \\ = \Theta\left[\partial_x + E, \partial_{t_{-n}}\right]\Theta^{-1} &= 0\,.\end{aligned} \tag{30}$$

In particular for $n = 1$ we find

$$\Big[\partial_x + E + A_0 + A_{1/2} + k_0, \partial_{t_{-1}} + D_1^{(-1)} + D_1^{(-1/2)}\Big] = 0\,, \tag{31}$$

where from (18)

$$D_1^{(-1)} = BE^{(-1)}B^{-1}\,, \quad D_1^{(-1/2)} = Bj_{-1/2}B^{-1}\,, \quad j_{-1/2} = \Big[m^{(1/2)}, E^{(-1)}\Big]\,.$$

Taking the grade -1 of eqn. (31) we find

$$\partial_x(BE^{(-1)}B^{-1}) + [A_0, BE^{(-1)}B^{-1}] = 0\,,$$

which has solution $A_0 = -\partial_x BB^{-1}$. Take now the grade $-\frac{1}{2}$ of eqn. (31) to obtain respectively

$$\partial_x\left(j_{-1/2}\right) = [E^{(-1)}, B^{-1}A_{1/2}B]\,. \tag{32}$$

From grades $\frac{1}{2}$ and zero, we obtain

$$\partial_{-1}\left(A_{1/2}\right) = \left[E, Bj_{-1/2}B^{-1}\right], \tag{33}$$

$$\partial_{-1}(\partial_x B\, B^{-1}) = \left[BE^{(-1)}B^{-1}, E\right] + \left[Bj_{-1/2}B^{-1}, A_{1/2}\right]. \tag{34}$$

Eqns. (32) – (34) correspond to the equations of motion of the associated relativistic integrable model.

6 The $N = 2$ super MKdV and sinh−Gordon equations

As an application consider the loop algebra $\widehat{SL}(2,2)$ described in the appendix. In parametrizing the Lax components $A_{1/2}$ and A_0 in (30) we determine from (26) and (25) the first two terms of Θ in (23), namely $\theta_{-1/2}$ and θ_{-1}. The lowest grade $\theta_{-1/2}$, in turn, determines the quantity k_0 from eqn. (27). From [11] we found that the existence of nontrivial k_0 gives rise to nonlocal supersymmetry transformation. In order to provide a simple example of local $N = 2$ supersymmetric integrable model we shall consider a subalgebra of the loop algebra $\widehat{SL}(2,2)$ whose generators are given by (59). Within such subalgebra we make sure that $k_0 = 0$. Let the Lax operator

$$L = \partial_x + E + A_0 + A_{1/2} \tag{35}$$

be specified by

$$E = K_1^{(1)} + K_2^{(1)} + I^{(1)}, \;\; A_0 = u_1 M_1^{(0)} + u_3 M_3^{(0)}, \;\; A_{1/2} = \bar{\psi}_1 G_1^{(1/2)} + \bar{\psi}_3 G_3^{(1/2)}. \tag{36}$$

We now solve the zero–curvature equation (12) for $n = 3$. It is explicitly given by

$$\begin{gathered}\left[\partial_x + E + A_0 + A_{1/2}, \partial_3 + \mathcal{D} + E^{(3)}\right] = 0\,,\\ \mathcal{D} = D_3^{(0)} + D_3^{(1/2)} + D_3^{(1)} + D_3^{(3/2)} + D_3^{(2)} + D_3^{(5/2)}\,,\end{gathered} \tag{37}$$

where $E^{(3)} = K_1^{(3)} + K_2^{(3)} + I^{(3)}$. We then obtain the following equations of motion for the $N = 2$ super MKdV,

$$4\partial_3\psi_1 = \partial_x^3\psi_1 - \tfrac{3}{2}\psi_1\partial_x(u_1^2 + u_3^2) - 3\partial_x(\psi_1)(u_1^2 + u_3^2) - 3\psi_3\partial_x(u_1u_3)\,, \tag{38}$$

$$4\partial_3\psi_3 = \partial_x^3\psi_3 - \tfrac{3}{2}\psi_3\partial_x(u_1^2 + u_3^2) - 3\partial_x(\psi_3)(u_1^2 + u_3^2) - 3\psi_1\partial_x(u_1u_3) \tag{39}$$

and

$$\begin{aligned}4\partial_3 u_1 &= \partial_x\Big[\partial_x^2 u_1 - 2u_1^3 + 3u_1(\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3) + 3u_3(-\psi_1\partial_x\psi_3 + \psi_3\partial_x\psi_1)\Big],\\ 4\partial_3 u_3 &= \partial_x\Big[\partial_x^2 u_3 - 2u_3^3 - 3u_3(\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3) - 3u_1(-\psi_1\partial_x\psi_3 + \psi_3\partial_x\psi_1)\Big].\end{aligned} \tag{40}$$

These equations are invariant under the following supersymmetry transformations

$$\delta u_1 = 2\partial_x(-\psi_1\epsilon_2 + \psi_3\epsilon_4)\,, \quad \delta u_2 = 2\partial_x(-\psi_1\epsilon_4 + \psi_3\epsilon_2) \tag{41}$$

and

$$\delta\psi_1 = u_1\epsilon_2 - u_3\epsilon_4\,, \quad \delta\psi_3 = u_1\epsilon_4 - u_3\epsilon_2\,, \tag{42}$$

with ϵ_2, ϵ_4 constant grassmanian parameters.

For the relativistic case we parametrize $B = \mathrm{e}^{\phi_1 M_1 + \phi_3 M_3}$ and

$$A_0 = -\partial_x B\, B^{-1} = -\partial_x \phi_1\, M_1 - \partial_x \phi_3 M_3\,, \tag{43}$$

$$A_{1/2} = \psi_1 G_1^{(1/2)} + \psi_3 G_3^{(1/2)}\,, \tag{44}$$

$$\jmath_{-1/2} = \psi_2 G_2^{(-1/2)} + \psi_4 G_4^{(-1/2)}\,, \tag{45}$$

$$E^{(-1)} = K_1^{(-1)} + K_2^{(-1)} + I^{(-1)}\,. \tag{46}$$

With notation

$$\bar\psi_\pm = \psi_2 \pm \psi_4\,, \quad \psi_\pm = \psi_1 \pm \psi_3\,, \quad \phi^\pm = \phi_1 \pm \phi_3 \tag{47}$$

the eqns. of motion (32)−(34) become

$$\partial_{-1}\psi_\pm = -2\bar\psi_\mp \cosh\phi_\pm\,, \tag{48}$$

$$\partial_x\bar\psi_\pm = -2\psi_\mp \cosh\phi_\pm\,, \tag{49}$$

$$\partial_{-1}\partial_x\phi_+ = -4\sinh\phi_+\cosh\phi_- + 4\psi_+\bar\psi_+\sinh\phi_-\,, \tag{50}$$

$$\partial_{-1}\partial_x\phi_- = -4\cosh\phi_+\sinh\phi_- + 4\psi_-\bar\psi_-\sinh\phi_+,. \tag{51}$$

These equations are invariant under the supersymmetry transformations:

$$\delta\phi_\pm = 2\psi_\mp\epsilon_\pm\,, \quad \delta\psi_\pm = -\partial_x\phi_\mp\epsilon_\pm\,, \quad \delta\bar\psi_\pm = 2\sinh\phi_\pm\epsilon_\mp\,, \tag{52}$$

where $\epsilon^\pm = \epsilon_2 \pm \epsilon_4$. The above $N = 2$ Sine–Gordon equations correspond to the ones proposed by Kobayashi and Uematsu [12] and written in the above form by Nepomechie [13]. The construction of integrable models with higher supersymmetries such as $N = 4$, 8 involves higher rank subalgebras of $\hat{SL}(4,4)$, $\hat{SL}(8,8)$ [14].

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Appendix: Algebra $SL(2,2)$

The super Lie algebra $SL(2,2)$ is a rank 3 algebra with simple roots

$$\alpha_1 = e_1 - e_2\,, \quad \alpha_2 = e_2 - f_1\,, \quad \alpha_3 = f_1 - f_2, \quad e_i\cdot e_j = -f_i\cdot f_j = \delta_{ij}\,. \tag{53}$$

The affine (loop) algebra $\widehat{SL}(2,2)$ is given by

$$\begin{aligned}
\left[\tilde h_i^{(n)}, E_{\alpha_j}^{(m)}\right] &= (\alpha_i\cdot\alpha_j)E_{\alpha_j}^{(n+m)}, && \\
\left[E_{\alpha_i}^{(n)}, E_{-\alpha_i}^{(m)}\right] &= \mathrm{Str}\,(E_{\alpha_i}, E_{-\alpha_i})\tilde h_i^{(n+m)}, && i,\, j = 1,\, 2,\, 3\,, \\
\left[E_{\alpha}^{(n)}, E_{\beta}^{(m)}\right] &= \epsilon(\alpha,\beta)E_{\alpha+\beta}^{(n+m)}, && \alpha+\beta \ \text{ is root}\,, \\
\left[E_{\alpha}^{(n)}, E_{\beta}^{(m)}\right] &= 0\,, && \text{otherwise}\,, \\
\left[d, T^{(m)}\right] &= mT^{(m)}, && T^{(m)} = E_\alpha^{(m)} \,\text{or}\, h_i^{(m)}\,,
\end{aligned} \tag{54}$$

where $n,m \in Z$ or $n,m \in Z+\frac{1}{2}$ according to the bosonic or fermionic character of the generator respectively. Define now

$$
\begin{aligned}
Q &= d + \tfrac{1}{2}(\tilde{h}_1^{(0)} + \tilde{h}_3^{(0)})\,,\\
E &= (E_{\alpha_1}^{(0)} + E_{-\alpha_1}^{(2)}) + (E_{\alpha_3}^{(2)} + E_{-\alpha_3}^{(0)}) + I^{(1)}\,.
\end{aligned}
$$

The fermionic and bosonic components of the Kernel $\mathcal{K}$ are generated by

$$
\begin{aligned}
f_{1,\eta}^{(n+1/2)} &= \left(\eta E_{\alpha_1+\alpha_2}^{(n-1/2)} + E_{-\alpha_1-\alpha_2}^{(n+3/2)}\right) + \left(\eta E_{\alpha_2+\alpha_3}^{(n+33/2)} + E_{-\alpha_2-\alpha_3}^{(n-1/2)}\right),\\
f_{2,\eta}^{(n+1/2)} &= \left(\eta E_{\alpha_1+\alpha_2+\alpha_3}^{(n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(n+1/2)}) + (\eta E_{\alpha_2}^{(n+1/2)} + E_{-\alpha_2}^{(n+1/2)}\right),
\end{aligned}
\tag{55}
$$

$\eta = \pm 1$ and

$$
\begin{aligned}
K_1^{(n)} &= E_{\alpha_1}^{(n-1)} + E_{-\alpha_1}^{(n+1)},\\
K_2^{(n)} &= E_{\alpha_3}^{(n+1)} + E_{-\alpha_3}^{(n-1)},\\
I^{(n)} &= \tilde{h}_1^{(n)} + 2\tilde{h}_2^{(n)} - \tilde{h}_3^{(n)}.
\end{aligned}
\tag{56}
$$

The image $\mathcal{M}$ by

$$
\begin{aligned}
g_{1,\eta}^{(n+1/2)} &= \left(\eta E_{\alpha_1+\alpha_2}^{(n-1/2)} + E_{-\alpha_1-\alpha_2}^{(n+3/2)}\right) - \left(\eta E_{\alpha_2+\alpha_3}^{(n+3/2)} + E_{-\alpha_2-\alpha_3}^{(n-1/2)}\right),\\
g_{2,\eta}^{(n+1/2)} &= \left(\eta E_{\alpha_1+\alpha_2+\alpha_3}^{(n+1/2)} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(n+1/2)}\right) - \left(\eta E_{\alpha_2}^{(n+1/2)} + E_{-\alpha_2}^{(n+1/2)}\right),
\end{aligned}
\tag{57}
$$

$\eta = \pm 1$ and

$$
\begin{aligned}
M_1^{(n)} &= \tilde{h}_1^{(n)}, \qquad M_2^{(n)} = -E_{\alpha_1}^{(n-1)} + E_{-\alpha_1}^{(n+1)},\\
M_3^{(n)} &= -\tilde{h}_3^{(n)}, \quad\; M_4^{(n)} = -E_{\alpha_3}^{(n-1)} + E_{-\alpha_3}^{(n+1)}
\end{aligned}
\tag{58}
$$

respectively. Define now

$$
\begin{aligned}
F_1^{(n+1/2)} &= \frac{1}{\sqrt{2}}\left(f_{1,+}^{(n+1/2)} + f_{2,+}^{(n+1/2)}\right), \quad F_2^{(n+1/2)} = \frac{1}{\sqrt{2}}\left(f_{1,-}^{(n+1/2)} + f_{2,-}^{(n+1/2)}\right),\\
F_3^{(n+1/2)} &= \frac{1}{\sqrt{2}}\left(f_{1,+}^{(n+1/2)} - f_{2,+}^{(n+1/2)}\right), \quad F_4^{(n+1/2)} = \frac{1}{\sqrt{2}}\left(f_{1,-}^{(n+1/2)} - f_{2,-}^{(n+1/2)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
G_1^{(n+1/2)} &= \frac{1}{\sqrt{2}}\left(g_{1,+}^{(n+1/2)} + g_{2,+}^{(n+1/2)}\right), \quad G_2^{(n+1/2)} = \frac{1}{\sqrt{2}}\left(g_{1,-}^{(n+1/2)} + g_{2,-}^{(n+1/2)}\right),\\
G_3^{(n+1/2)} &= \frac{1}{\sqrt{2}}\left(g_{1,+}^{(n+1/2)} - g_{2,+}^{(n+1/2)}\right), \quad G_4^{(n+1/2)} = \frac{1}{\sqrt{2}}\left(g_{1,-}^{(n+1/2)} - g_{2,-}^{(n+1/2)}\right).
\end{aligned}
$$

We now define a consistent subalgebra of the affine $\widehat{SL}(2,2)$ loop algebra by selecting the following generators,

$$
\begin{aligned}
&M_1^{(2n)},\ M_3^{(2n)},\ M_2^{(2n+1)},\ M_4^{(2n+1)},\ K_1^{(2n+1)},\ K_2^{(2n+1)},\ I^{(2n+1)},\\
&G_1^{(2n+1/2)},\ G_3^{(2n+1/2)},\ F_2^{(2n+1/2)},\ F_4^{(2n+1/2)},\\
&G_2^{(2n+3/2)},\ G_4^{(2n+3/2)},\ F_1^{(2n+3/2)},\ F_3^{(2n+3/2)} \qquad\qquad \text{for } n \in Z.
\end{aligned}
\tag{59}
$$

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