Quasi−exact solvability of the Dirac equations

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We present a general procedure for determining quasi–exact solvability of the Dirac and the Pauli equation with an underlying $sl(2)$ symmetry. This procedure makes full use of the close connection between quasi–exactly solvable systems and supersymmetry.

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In this talk we presented a general procedure for determining quasi–exact solvability of the Dirac and the Pauli equation with an underlying $sl(2)$ symmetry. This procedure makes full use of the close connection between quasi–exactly solvable systems and supersymmetry (SUSY). Based on this procedure, we demonstrate that the Pauli equation, the Dirac equation coupled minimally with a vector potential, and neutral Dirac particles in external electric fields, which are equivalent to generalized Dirac oscillators, are physical examples of quasi–exactly solvable systems.

Here we only give the main ideas of the procedures, and refer the readers to [1, 2] for details.

For all the cases cited above, one can reduce the corresponding multi–component equations to a set of one–variable equations possessing one-dimensional SUSY after separating the variables in a suitable coordinate system. Typically the set of equations takes the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+W(r)\right)f_- = \mathcal{E}^+ f_+\,, \tag{1}$$

$$\left(-\frac{\mathrm{d}}{\mathrm{d}r}+W(r)\right)f_+ = \mathcal{E}^- f_-\,, \tag{2}$$

where r is the basic variable, e.g. the radial coordinate, and $f_\pm$ are, say, the two components of the radial part of the Dirac wave function. The superpotential W is related to the external field configuration, and $\mathcal{E}^\pm$ involve the energy and mass of the particle. We can rewrite this set of equations as

$$A^-A^+f_- = \epsilon f_-\,, \tag{3}$$

$$A^+A^-f_+ = \epsilon f_+\,, \tag{4}$$

with

$$A^\pm \equiv \pm\frac{\mathrm{d}}{\mathrm{d}r}+W\,,\quad \epsilon\equiv\mathcal{E}^+\mathcal{E}^-\,. \tag{5}$$

Explicitly, the above equations read

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+W^2\mp W'\right)f_\mp = \epsilon f_\mp\,. \tag{6}$$

Here and below the prime means differentiation with respect to the basic variable. Eq.(6) clearly exhibits the SUSY structure of the system. The operators acting on $f_\pm$ in eq.(6) are said to be factorizable, i.e. as products of A^- and A^+. The ground state, with $\epsilon = 0$, is given by one of the following two sets of equations:

$$A^+ f_-^{(0)}(r) = 0\,, \qquad f_+^{(0)}(r) = 0\,; \tag{7}$$

$$A^- f_+^{(0)}(r) = 0\,, \qquad f_-^{(0)}(r) = 0\,, \tag{8}$$

depending on which solution is normalizable.

One can determine the forms of the external field that admit exact solutions of the problem by comparing the forms of the superpotential W with those listed in Table (4.1) of [3].

Similarly, from Turbiner's classification of the $sl(2)$ QES systems [4], one can determine the forms of W, and hence the forms of external fields admitting QES solutions based on $sl(2)$ algebra. The main ideas of the procedures are outlined below. We shall concentrate only on solution of the upper component f_-, which is assumed to have a normalizable zero energy state.

Eq.(6) shows that f_- satisfies the Schrödinger equation $H_- f_- = \epsilon f_-$, with

$$H_- = A^- A^+ = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r)\,, \tag{9}$$

with

$$V(r) = W(r)^2 - W'(r)\,. \tag{10}$$

We shall look for $V(r)$ such that the system is QES. According to the theory of QES models, one first makes an "imaginary gauge transformation" on the function f_-

$$f_-(r) = \phi(r)\mathrm{e}^{-g(r)}\,, \tag{11}$$

where $g(r)$ is called the gauge function. The function $\phi(r)$ satisfies

$$-\frac{\mathrm{d}^2\phi(r)}{\mathrm{d}r^2} + 2g'\frac{\mathrm{d}\phi(r)}{\mathrm{d}r} + \left[V(r) + g'' - g'^2\right]\phi(r) = \epsilon\phi(r)\,. \tag{12}$$

For physical systems which we are interested in, the phase factor $\exp\bigl(-g(r)\bigr)$ is responsible for the asymptotic behaviors of the wave function so as to ensure normalizability. The function $\phi(r)$ satisfies a Schrödinger equation with a gauge transformed Hamiltonian

$$H_G = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + 2W_0(r)\frac{\mathrm{d}}{\mathrm{d}r} + \left[V(r) + W_0' - W_0^2\right]\,, \tag{13}$$

where $W_0(r) = g'(r)$. Now if $V(r)$ is such that the quantal system is QES, that means the gauge transformed Hamiltonian H_G can be written as a quadratic combination of the generators J^a of some Lie algebra with a finite dimensional representation. Within this finite dimensional Hilbert space the Hamiltonian H_G can be diagonalized, and therefore a finite number of eigenstates are solvable. For one–dimensional QES systems the most general Lie algebra is $sl(2)$. Hence if eq.(13) is QES then it can be expressed as

$$H_G = \sum C_{ab} J^a J^b + \sum C_a J^a + \text{constant}\,, \tag{14}$$

where C_{ab}, C_a are constant coefficients, and the J^a are the generators of the Lie algebra $sl(2)$ given by

$$\begin{aligned} J^+ &= z^2 \frac{\mathrm{d}}{\mathrm{d}z} - Nz\,, \\ J^0 &= z\frac{\mathrm{d}}{\mathrm{d}z} - \frac{N}{2}\,, \quad N = 0, 1, 2, \ldots\,, \\ J^- &= \frac{\mathrm{d}}{\mathrm{d}z}\,. \end{aligned} \tag{15}$$

Here the variables r and z are related by $z = h(r)$, where $h(\cdot)$ is some (explicit or implicit) function. The value $j = N/2$ is called the weight of the differential representation of $sl(2)$ algebra, and N is the degree of the eigenfunctions ϕ, which are polynomials in a $(N+1)$–dimensional Hilbert space

$$\phi = (z - z_1)(z - z_2)\cdots(z - z_N)\,. \tag{16}$$

The requirement in eq.(14) fixes $V(r)$ and $W_0(r)$, and H_G will have an algebraic sector with $N+1$ eigenvalues and eigenfunctions. For definiteness, we shall denote the potential V admitting $N+1$ QES states by V_N. From eqs.(11) and (16), the function f_- in this sector has the general form

$$f_- = (z - z_1)(z - z_2)\cdots(z - z_N)\exp\left(-\int^z W_0(r)\,\mathrm{d}r\right), \tag{17}$$

where z_i $(i = 1, 2, \ldots, N)$ are N parameters that can be determined by plugging eq.(17) into eq.(12). The algebraic equations so obtained are called the Bethe ansatz equations corresponding to the QES problem [5, 1, 2]. Now one can rewrite eq.(17) as

$$f_- = \exp\left(-\int^z W_N(r, \{z_i\})\,\mathrm{d}r\right), \tag{18}$$

with

$$W_N(r, \{z_i\}) = W_0(r) - \sum_{i=1}^{N} \frac{h'(r)}{h(r) - z_i}\,. \tag{19}$$

There are $N+1$ possible functions $W_N(r, \{z_i\})$ for the $N+1$ sets of eigenfunctions ϕ. Inserting eq.(18) into $H_- f_- = \epsilon f_-$, one sees that W_N satisfies the Ricatti equation

$$W_N^2 - W_N' = V_N - \epsilon_N\,, \tag{20}$$

where ϵ_N is the energy parameter corresponding to the eigenfunction f_- given in eq.(17) for a particular set of N parameters $\{z_i\}$.

From eqs.(9), (10) and (20) it is clear how one should proceed to determine the external fields so that the Dirac equation becomes QES based on $sl(2)$: one needs only to determine the superpotentials $W(r)$ according to eq.(20) from the QES potentials $V(r)$ classified in [4]. This is easily done by observing that the superpotential W_0 corresponding to $N = 0$ is related to the gauge function $g(r)$ associated with a particular class of QES potential $V(r)$ by $g'(r) = W_0(r)$. This superpotential gives the field configuration that allows the weight zero ($j = N = 0$) state, i.e. the ground state, to be known in that class. The more interesting task is to obtain higher weight states (i.e. $j > 0$), which will include excited states. For weight j ($N = 2j$) states, this is achieved by forming the superpotential $W_N(r, \{z_i\})$ according to eq.(19). Of the $N + 1$ possible sets of solutions of the Bethe ansatz equations, the set of roots $\{z_1, z_2, \ldots, z_N\}$ to be used in eq.(19) is chosen to be the set for which the energy parameter of the corresponding state is the lowest (usually it is the ground state).

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