Symmetries and graded contractions of the Pauli graded $sl(3,\mathbb{C})$

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Presented results were achieved in collaboration with Miloslav Havlíček, Jiří Patera and Jiří Tolar. We consider the Pauli grading of the Lie algebra $sl(3,\mathbb{C})$ and use a concept of graded contractions to construct non-isomorphic Lie algebras of dimension 8, while preserving the Pauli grading. We show how the symmetry group of a grading simplifies the solution of contraction equations and identification of results. We give examples of resulting Lie algebras. Complete results will be published elsewhere.

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1 Introduction

Symmetry groups belong to the main tools in theoretical physics. Among them, special role belongs to Lie groups or groups of continuous symmetries due to the importance of Noether's theorem allowing to relate symmetries and conserved quantities in field theories. Instead of Lie group action it is usually sufficient to consider infinitesimal symmetry transformations induced by the corresponding Lie algebra.

In the last decade important results were obtained in the classification of gradings of classical simple Lie algebras. For a given simple Lie algebra it is thus possible — with some effort — to determine all its gradings. Especially the fine gradings are useful because other gradings are obtained from them by suitable combinations of theirs grading subspaces. On the mathematical side, fine gradings of simple Lie algebras are analogous of Cartan's root decomposition. They define new bases with unexpected properties. On the physical side, they yield quantum observables with additive quantum numbers.

Relations between Lie algebras of the same dimension can be studied by means of contractions or deformations. Special place belongs to graded contractions — contractions which preserve a given grading and in this way yield further Lie algebras with the same additive quantum numbers.

Since the classification of complex and real Lie algebras of low dimensions 3, 4 and 5 is known, the obtained graded contractions in these dimensions are classified. This is the case e. g. for graded contractions of $A_1 = sl(2,\mathbb{C})$ and its real forms in [17].

However a single step to $A_2 = sl(3,\mathbb{C})$ leads us to an unexplored territory of 8–dimensional complex Lie algebras. Besides the root decomposition, A_2 has 3 fine gradings, and the full graph of its gradings consists of 17 gradings including the 4

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fine gradings and the trivial grading — the whole A_2. In order to obtain all graded contractions of A_2, it is sufficient to consider only the fine gradings, since among them all contractions are found which would come from the coarse gradings.

Graded contractions for the root decomposition were obtained in the papers [2],[1]. The investigations resulted there in 32 Lie algebras, of which 9 are 8–dimensional non-decomposable. The present study is devoted to graded contractions of A_2 for so-called Pauli grading. It seems that the results are most numerous among all four fine gradings.

2 Gradings and graded contractions of Lie algebras

2.1 Lie gradings

Let us consider a finite–dimensional Lie algebra $\mathcal{L}$ over the field of complex numbers $\mathbb{C}$. A decomposition of this algebra into a direct sum of its subspaces $\mathcal{L}_i, i \in I$

$$\mathcal{L} = \bigoplus_{i\in I} \mathcal{L}_i \tag{1}$$

is called a **grading** of Lie algebra $\mathcal{L}$, when the following property holds

$$(\forall i, j \in I)(\exists k \in I)([\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_k)\,, \tag{2}$$

where I is an index set, and we denoted $[\mathcal{L}_i, \mathcal{L}_j] := \big\{[x, y]\big| x \in \mathcal{L}_i, y \in \mathcal{L}_j\big\}$. Subspaces $\mathcal{L}_i, i \in I$ are called **grading subspaces**. Grading $\Gamma : \mathcal{L} = \bigoplus_{i\in I} \mathcal{L}_i$ is a **refinement** of grading $\tilde{\Gamma} : \mathcal{L} = \bigoplus_{j\in J} \tilde{\mathcal{L}}_j$ if for each $i \in I$ there exists $j \in J$ such that $\mathcal{L}_i \subseteq \tilde{\mathcal{L}}_j$. Refinement is called **proper** if the cardinality of I is greater than the cardinality of the set J. Grading which cannot be properly refined is called **fine**. If all grading subspaces are one–dimensional, then the grading is called **finest**. The property (2) defines a binary operation on the set I. If $[\mathcal{L}_i, \mathcal{L}_j] = \{0\}$ holds, we can choose an arbitrary k. It is proved in [15] that for simple Lie algebras the index set I with this operation can always be embedded into an *Abelian group* G; we are going to denote the operation additively and we have

$$[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}\,, \quad \text{where } i, j, i + j \in G\,. \tag{3}$$

Then we say that the Lie algebra is graded by the group G, or it is ***G*−graded**. Group G is called a **grading group**.

Gradings of a Lie algebra $\mathcal{L}$ are closely related to the group of automorphisms $\operatorname{Aut}\mathcal{L}$. Automorphisms induce equivalence of gradings: if $\Gamma : \mathcal{L} = \bigoplus_{i\in I} \mathcal{L}_i$ is a grading of $\mathcal{L}$, then for an arbitrary automorphism $g \in \operatorname{Aut}\mathcal{L}$ $\tilde{\Gamma} : \mathcal{L} = \bigoplus_{i\in I} g(\mathcal{L}_i)$ is also a grading of $\mathcal{L}$. Such gradings Γ and $\tilde{\Gamma}$ are called **equivalent**. The maximal set of diagonable and mutually commuting automorphisms is a subgroup of $\operatorname{Aut}\mathcal{L}$ called a **MAD−group** (**m**aximal **A**belian group of **d**iagonable automorphisms). Each given grading (2) determines a subgroup $\operatorname{Diag}\Gamma \subset \operatorname{Aut}\mathcal{L}$ containing all automorphisms $g \in GL(\mathcal{L})$, which preserve Γ, $g(\mathcal{L}_i) = \mathcal{L}_i$, and are **diagonal**,

$$gx = \lambda_i x \quad \text{for all } x \in \mathcal{L}_i,\ i \in I\,,$$

where $\lambda_i \neq 0$ depends only on g and $i \in I$. In [15] an important theorem was proved which, for all simple Lie algebras, is the basis for classification of all their possible fine gradings.

Theorem 2.1. Let $\mathcal{L}$ be a finite–dimensional simple Lie algebra over an algebraically closed field of characteristic zero. Then the grading Γ is fine if and only if $\mathrm{Diag}\,\Gamma$ is equal to some MAD–group.

In this way, the problem of classification of all fine gradings of simple Lie algebras is converted into the classification of all MAD–groups in $\mathrm{Aut}\,\mathcal{L}$. This task was studied in papers [7], [8] and [5].

2.2 The Pauli gradings of $sl(n,\mathbb{C})$

Let us first consider Lie algebras $sl(n,\mathbb{C})$ in general. We introduce the following notation: subgroup in $GL(n,\mathbb{C})$ containing all regular diagonal matrices is denoted $D(n)$. We also define $n \times n$ matrices

$$Q_n = \mathrm{diag}\,(1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1})\,, \tag{4}$$

where $\omega_n = \exp(2\pi \mathrm{i}/n)$, and matrix

$$P_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{5}$$

For $n = 1$ we set $Q_1 = P_1 = (1)$. The grading Γ_n of $sl(n,\mathbb{C})$ defined as

$$\Gamma_n : \, sl(n,\mathbb{C}) = \bigoplus_{(r,s)\in\mathbb{Z}_n\times\mathbb{Z}_n\setminus(0,0)} \mathcal{L}_{rs}\,, \tag{6}$$

where $\mathcal{L}_{rs} := \{X_{rs}\}_{lin}$ and

$$X_{rs} = {Q_n}^r {P_n}^s \tag{7}$$

is called the **Pauli grading**. It is, in fact, finest, i.e. all $n^2 - 1 = \dim sl(n,\mathbb{C})$ subspaces are one–dimensional. We can easily check that (6) is indeed a grading by verification of the property (3):

$$[X_{rs}, X_{r's'}] = (\omega_n^{sr'} - \omega_n^{rs'}) X_{r+r',s+s'}\,. \tag{8}$$

Hence we see that our grading group G is equal to the additive Abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$ with addition componentwise (modulo n).

Since this article focuses on the Pauli grading of $sl(3,\mathbb{C})$ we list its explicit formula arising from (6), (7):

$$\begin{aligned}\Gamma_3 : sl(3,\mathbb{C}) &= \mathcal{L}_{01} \oplus \mathcal{L}_{02} \oplus \mathcal{L}_{10} \oplus \mathcal{L}_{20} \oplus \mathcal{L}_{11} \oplus \mathcal{L}_{22} \oplus \mathcal{L}_{12} \oplus \mathcal{L}_{21} = \\ &= Q_3 \oplus Q_3^2 \oplus P_3 \oplus P_3^2 \oplus P_3Q_3 \oplus P_3^2Q_3^2 \oplus P_3Q_3^2 \oplus P_3^2Q_3 = \\ &= \begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix} \oplus \begin{pmatrix}1&0&0\\0&\omega^2&0\\0&0&\omega\end{pmatrix} \oplus \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \oplus \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \oplus \\ &\quad \oplus \begin{pmatrix}0&\omega&0\\0&0&\omega^2\\1&0&0\end{pmatrix} \oplus \begin{pmatrix}0&0&\omega\\1&0&0\\0&\omega^2&0\end{pmatrix} \oplus \begin{pmatrix}0&\omega^2&0\\0&0&\omega\\1&0&0\end{pmatrix} \oplus \begin{pmatrix}0&0&\omega^2\\1&0&0\\0&\omega&0\end{pmatrix},\end{aligned} \tag{9}$$

where $\omega = \exp(2\pi \mathrm{i}/3)$. Note that the symbol for linear span was omitted.

2.3 Graded contractions of Lie algebras

Let us state the definition of a graded contraction. Suppose $\mathcal{L}$ is a Lie algebra graded by a group G, i.e. the relations (1) and (3) hold. Introducing **contraction parameters** ε_{ij} we define a bilinear mapping $[\ ,\ ]_\varepsilon$ on $\mathcal{L}$ (more precisely on the underlying vector space V) by the formula

$$[x,y]_\varepsilon := \varepsilon_{ij}[x,y] \quad \text{for all} \quad x \in \mathcal{L}_i,\ y \in \mathcal{L}_j,\ i,j \in I \text{ and } \varepsilon_{ij} \in \mathbb{C}\,. \tag{10}$$

Since we claim the bilinearity of $[\ ,\ ]_\varepsilon$, the condition (10) determines this mapping on the whole V. Note that if the subspaces $\mathcal{L}_i$, $\mathcal{L}_j$ commute, $[\mathcal{L}_i, \mathcal{L}_j] = \{0\}$, then $[\mathcal{L}_i, \mathcal{L}_j]_\varepsilon = \{0\}$ as well, independently of ε_{ij}; we call such positions **irrelevant** and set these ε_{ij} equal to zeros. If $\mathcal{L}^\varepsilon := (V, [\ ,\ ]_\varepsilon)$ is a Lie algebra, then it is called a **graded contraction** of the Lie algebra $\mathcal{L}$. The graded contraction preserves a grading because it is also true that $\mathcal{L}^\varepsilon = \bigoplus_{i\in G} \mathcal{L}_i$ is a grading of $\mathcal{L}^\varepsilon$. There are two conditions which the parameters ε_{ij} must fulfill for $\mathcal{L}^\varepsilon$ to become a Lie algebra: antisymmetry of $[\ ,\ ]_\varepsilon$ immediately gives

$$\varepsilon_{ij} = \varepsilon_{ji}\,, \tag{11}$$

hence each such solution can be written as a *symmetric* matrix $\varepsilon = (\varepsilon_{ij})$ which is called a **contraction matrix**. The validity of the Jacobi identity is the second condition and it is equivalent to the following property: for all (unordered) triples $i,j,k \in I$

$$e(i\,j\,k) :\ [x,[y,z]_\varepsilon]_\varepsilon + [z,[x,y]_\varepsilon]_\varepsilon + [y,[z,x]_\varepsilon]_\varepsilon = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k) \tag{12}$$

is satisfied. Each $e(i\,j\,k)$, $i,j,k \in I$ is then called a **contraction equation**. We call the set of contraction equations a **contraction system** $\mathcal{S}$, and the set of its solutions is denoted $\mathcal{R}(\mathcal{S})$.

Let us introduce normalization process for contraction matrices. At first we introduce a **commutative elementwise matrix multiplication** $\bullet$, i.e. for two matrices $A=(A_{ij})$, $B=(B_{ij})$ we define a matrix $C:=(C_{ij})$ by the formula

$$C_{ij} := A_{ij}B_{ij} \quad \text{(no summation)} \tag{13}$$

and write $C = A \bullet B$. For the given grading (1) we also define a matrix $\alpha := (\alpha_{ij})$, where

$$\alpha_{ij} = \frac{a_i a_j}{a_{i+j}} \quad \text{for } i,j \in I \tag{14}$$

and $a_i \in \mathbb{C}\setminus\{0\}$ for all $i \in I$. The matrix α is then called a **normalization matrix**. Normalization is a process based on the following lemma:

Lemma 2.2. Let $\mathcal{L}^\varepsilon$ be a graded contraction of a graded Lie algebra $\mathcal{L}=\bigoplus_{i\in I}\mathcal{L}_i$. Then $\mathcal{L}^{\widetilde{\varepsilon}}$, where $\widetilde{\varepsilon} := \alpha\bullet\varepsilon$, is for any normalization matrix α a graded contraction of $\mathcal{L}$ and the Lie algebras $\mathcal{L}^{\widetilde{\varepsilon}}$ and $\mathcal{L}^\varepsilon$ are isomorphic, $\mathcal{L}^{\widetilde{\varepsilon}} \simeq \mathcal{L}^\varepsilon$.

Proof. We define a diagonal regular linear mapping $h \in GL(\mathcal{L})$ corresponding to a normalization matrix $\alpha = \left(\frac{a_i a_j}{a_{i+j}}\right)$ by the formula

$$hx_i = a_i x_i \quad i\in I,\, x_i \in \mathcal{L}_i\,. \tag{15}$$

Then for all $x\in\mathcal{L}_i$, $y\in\mathcal{L}_j$ the bilinear mapping $[x,y]_{\widetilde{\varepsilon}} = \widetilde{\varepsilon}_{ij}[x,y]$ and the Lie bracket $[x,y]_\varepsilon = \varepsilon_{ij}[x,y]$ satisfy

$$[x,y]_{\widetilde{\varepsilon}} = \widetilde{\varepsilon}_{ij}[x,y] = \frac{a_i a_j}{a_{i+j}}\varepsilon_{ij}[x,y] = h^{-1}[hx,hy]_\varepsilon\,.$$

Hence $\mathcal{L}^{\widetilde{\varepsilon}}$ is a Lie algebra and h is an isomorphism between $\mathcal{L}^{\widetilde{\varepsilon}}$ and $\mathcal{L}^\varepsilon$. ☐

3 Symmetries and graded contractions

3.1 Symmetry group of a grading

We define a **symmetry group** $\operatorname{Aut}\Gamma$ of a grading $\Gamma:\mathcal{L}=\bigoplus_{i\in I}\mathcal{L}_i$ as a subgroup of $\operatorname{Aut}\mathcal{L}$ which contains automorphisms g with the property $g\mathcal{L}_i = \mathcal{L}_{\pi_g(i)}$, where $\pi_g : I\to I$ is a permutation of the index set I. Thus, a permutation representation Δ_Γ of the group $\operatorname{Aut}\Gamma$ is given on the set I, defined as $\Delta_\Gamma(g) := \pi_g$. The kernel of this representation is a **stabilizer** of Γ in $\operatorname{Aut}\Gamma$,

$$\operatorname{Stab}\Gamma = \ker\Delta_\Gamma = \{g\in\operatorname{Aut}\mathcal{L} \mid g\mathcal{L}_i = \mathcal{L}_i \;\; \forall i\in I\}\,. \tag{16}$$

Hence the stabilizer is a normal subgroup of $\operatorname{Aut}\Gamma$ and we have

$$\operatorname{Aut}\Gamma/\operatorname{Stab}\Gamma \simeq \Delta_\Gamma\operatorname{Aut}\Gamma\,. \tag{17}$$

This permutation group $\Delta_\Gamma\operatorname{Aut}\Gamma$ is crucial for solving the system $\mathcal{S}$. It can be determined using relation (17). For fine gradings corresponding to the MAD-group $\mathcal{G}$ we have $\operatorname{Stab}\Gamma=\mathcal{G}$. We define a **normalizer** of a MAD–group $\mathcal{G}$ as a subgroup

$$\mathcal{N}(\mathcal{G}) = \left\{h\in\operatorname{Aut}\mathcal{L} \mid h^{-1}\mathcal{G}h \subset \mathcal{G}\right\}. \tag{18}$$

As shown in [6], we have

$$\mathcal{N}(\mathcal{G})/\mathcal{G} \simeq \Delta_\Gamma\operatorname{Aut}\Gamma\,. \tag{19}$$

3.2 Action of a symmetry group

Let us define a set of **relevant unordered pairs of grading indices** $\mathcal{I}$ by

$$\mathcal{I} := \left\{ i\,j \,\middle|\, i,j \in I,\ [\mathcal{L}_i, \mathcal{L}_j] \neq \{0\} \right\}, \tag{20}$$

where i, j denotes an unordered pair. For the Pauli grading of $sl(n,\mathbb{C})$ this set will be denoted by $\mathcal{I}_n$. Analyzing relations (8), we obtain explicitly

$$\mathcal{I}_n = \left\{ (ij)(kl) \,\middle|\, jk - il \neq 0 \,(\mathrm{mod}\ n),\ (ij),(kl) \in \mathcal{Z}_n \times \mathcal{Z}_n \setminus \{(0,0)\} \right\}. \tag{21}$$

A set of **relevant contraction parameters** ε_{ij}, due to (11), can be written as $\mathcal{E} := \{\varepsilon_k,\ k \in \mathcal{I}\}$. For a permutation $\pi \in \Delta_\Gamma \operatorname{Aut}\Gamma$ and a contraction matrix $\varepsilon = (\varepsilon_{ij})$, an **action of $\boldsymbol{\pi}$ on a contraction matrix** $\varepsilon \mapsto \varepsilon^\pi$ is defined by

$$(\varepsilon^\pi)_{ij} := \varepsilon_{\pi(i)\pi(j)}\,. \tag{22}$$

We observe that an **action on variables** $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ is in fact the action on the set of relevant variables $\mathcal{E}$: if $\varepsilon_{ij} \in \mathcal{E}$, $[\mathcal{L}_i, \mathcal{L}_j] \neq \{0\}$ and $g \in \operatorname{Aut}\Gamma$, $\Delta_\Gamma(g) = \pi$, then $\{0\} \neq g[\mathcal{L}_i, \mathcal{L}_j] = [\mathcal{L}_{\pi(i)}, \mathcal{L}_{\pi(j)}]$ and $\varepsilon_{\pi(i)\pi(j)} \in \mathcal{E}$. Hence the transformed matrix ε^π has zeros on the same irrelevant positions as the matrix ε.

Lemma 3.1. Let $\mathcal{L}^\varepsilon$ be a graded contraction of a graded Lie algebra $\Gamma : \mathcal{L} = \bigoplus_{i\in I} \mathcal{L}_i$. Then $\mathcal{L}^{\varepsilon^\pi}$ is, for any permutation $\pi \in \Delta_\Gamma$ Aut Γ, a graded contraction of $\mathcal{L}$ and the Lie algebras $\mathcal{L}^{\varepsilon^\pi}$ and $\mathcal{L}^\varepsilon$ are isomorphic, $\mathcal{L}^{\varepsilon^\pi} \simeq \mathcal{L}^\varepsilon$.

Proof. For given $\pi \in \Delta_\Gamma$ Aut Γ we take any $g \in \operatorname{Aut}\Gamma$ such that $\Delta_\Gamma(g) = \pi$. Consider

$$gx = z\,, \quad gy = w \quad x \in \mathcal{L}_i\,,\ y \in \mathcal{L}_j\,,\ z \in \mathcal{L}_{\pi(i)}\,,\ w \in \mathcal{L}_{\pi(j)}\,,\ i,j \in I\,. \tag{23}$$

Then for all $x \in \mathcal{L}_i$, $y \in \mathcal{L}_j$ the bilinear mapping $[x,y]_{\varepsilon^\pi} = \varepsilon_{\pi(i)\pi(j)}[x,y]$ and the Lie bracket $[x,y]_\varepsilon = \varepsilon_{ij}[x,y]$ satisfy

$$[x,y]_{\varepsilon^\pi} = \varepsilon_{\pi(i)\pi(j)}[x,y] = \varepsilon_{\pi(i)\pi(j)} g^{-1}[z,w] = g^{-1}[gx,gy]_\varepsilon\,. \tag{24}$$

Hence $\mathcal{L}^{\varepsilon^\pi}$ is a Lie algebra and g is an isomorphism between $\mathcal{L}^{\varepsilon^\pi}$ and $\mathcal{L}^\varepsilon$. □

3.3 Symmetries of the contraction system

Lemma 3.1 says that for a given contraction matrix ε it is possible to construct new contraction matrices ε^π, $\pi \in \Delta_\Gamma \operatorname{Aut}\Gamma$ which have to be solutions of the contraction system (and correspond to isomorphic Lie algebras). We thus obtained the substitutions $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$, $\pi \in \Delta_\Gamma \operatorname{Aut}\Gamma$ under which the set of solutions of the contraction system is invariant. Now we can also define an **action of $\boldsymbol{\Delta_\Gamma}$ Aut $\boldsymbol{\Gamma}$ on the contraction system $\boldsymbol{\mathcal{S}}$**: each equation in $\mathcal{S}$ is labeled by a triple of grading indices and we write $e(i\,j\,k) \in \mathcal{S}$ in the form

$$e(i\,j\,k) : \ [x,[y,z]_\varepsilon]_\varepsilon + \text{cyclically} = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k)\,; \tag{25}$$

then for each $\pi \in \Delta_\Gamma \operatorname{Aut}\Gamma$ we define the action

$$e(i\,j\,k) \;\mapsto\; e(\pi(i)\,\pi(j)\,\pi(k))\,. \tag{26}$$

Note that equation $e(\pi(i)\,\pi(j)\,\pi(k))$ can be written as

$$e(\pi(i)\,\pi(j)\,\pi(k)) : [gx,[gy,gz]_\varepsilon]_\varepsilon + \text{cyclically} = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k)\,, \tag{27}$$

where $g \in \operatorname{Aut}\Gamma$, $\Delta_\Gamma(g) = \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma$. According to (24) this is equal to

$$g[x,[y,z]_{\varepsilon^\pi}]_{\varepsilon^\pi} + \text{cyclically} = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k) \tag{28}$$

and (28) is satisfied if and only if

$$[x,[y,z]_{\varepsilon^\pi}]_{\varepsilon^\pi} + \text{cyclically} = 0 \quad (\forall x \in \mathcal{L}_i)(\forall y \in \mathcal{L}_j)(\forall z \in \mathcal{L}_k)\,. \tag{29}$$

Now equation (29) is precisely the equation (25) after the substitution $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$. In this way we have not only verified the invariance of the contraction system, but also have shown the method of its construction. Having chosen a starting equation one can write a whole orbit of equations by substituting $\varepsilon_{ij} \mapsto \varepsilon_{\pi(i)\pi(j)}$ till all $\pi \in \Delta_\Gamma \operatorname{Aut}\Gamma$ are exhausted. If we denote unordered k–tuple of grading indices $i_1, i_2, \ldots, i_k \in I$ as $i_1\, i_2 \ldots i_k$ and define an **action of $\Delta_\Gamma \operatorname{Aut}\Gamma$ on unordered $k-$tuples** as

$$i_1\, i_2\, \ldots i_k \;\mapsto\; \pi(i_1)\,\pi(i_2)\ldots\pi(i_k), \quad \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma\,, \tag{30}$$

then orbits of equations correspond to orbits of unordered triples of grading indices.

3.4 Equivalence of solutions

Combining Lemma 2.2 and Lemma 3.1 we define two solutions $\varepsilon', \varepsilon \in \mathcal{R}(\mathcal{S})$ to be **equivalent**, $\varepsilon' \sim \varepsilon$, if there exists a normalization matrix α and $\pi \in \Delta_\Gamma \operatorname{Aut}\Gamma$ such that

$$\varepsilon' = \alpha \bullet \varepsilon^\pi\,. \tag{31}$$

If two solutions are equivalent there exist a diagonal mapping $h \in GL(V)$ defined via formula (15) and an automorphism $g \in \operatorname{Aut}\Gamma$, $\Delta_\Gamma(g) = \pi$ with the property (24) satisfying

$$gh[x,y]_{\varepsilon'} = gh[x,y]_{\alpha\bullet\varepsilon^\pi} = g[hx,hy]_{\varepsilon^\pi} = [ghx,ghy]_\varepsilon$$

and *vice versa*, i.e. two solution ε', ε are equivalent, $\varepsilon' \sim \varepsilon$, if and only if

$$gh[x,y]_{\varepsilon'} = [ghx,ghy]_\varepsilon \tag{32}$$

holds for all $x \in \mathcal{L}_i$, $y \in \mathcal{L}_j$, $i,j \in I$. Let us note that (32) gives

Proposition 3.2. *Graded contractions corresponding to equivalent solutions are isomorphic.*

We introduce a finite matrix group

$$H_n = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \,\middle|\, a,b,c,d \in \mathbb{Z}_n,\ ad-bc = \pm 1 \bmod n \right\}. \tag{33}$$

This group is described in detail in [6]. It contains the subgroup of matrices with determinant +1 called $SL(2,\mathcal{Z}_n)$. An important theorem was proved in [6]:

Theorem 3.3. *The symmetry group $\Delta_{\Gamma_n}\,\mathrm{Aut}\,\Gamma_n$ of the Pauli grading of $sl(n,\mathbb{C})$ is isomorphic to the matrix group H_n. Denoting by π_A the permutation corresponding to the matrices $A \in H_n$, the action of π_A on the indices of the grading group $\mathcal{Z}_n \times \mathcal{Z}_n$ is given by*

$$\pi_A(i\,j) = (i\,j)A\,, \tag{34}$$

where matrix multiplication modulo n is found on the right hand side.

4 Contraction system for the Pauli grading of $sl(3,\mathbb{C})$

We denote the contraction system for the Pauli grading of $sl(n,\mathbb{C})$ by $\mathcal{S}_n$. This section is devoted to the contraction system $\mathcal{S}_3$. Let us take the Pauli grading in the form (9), the grading group is $\mathcal{Z}_3 \times \mathcal{Z}_3$; no subspace is labeled by $(0,0)$. Let us state the explicit form of matrix ε defined by (10); it is an 8×8 symmetric matrix and we will order the indices as in formula (9), i.e. positions 11, 12, 13, … are denoted (01)(01), (01)(02), (01)(10), … . Hence the symmetric contraction matrix ε with 24 relevant variables is of the following explicit form

$$\varepsilon = \begin{pmatrix}
0 & 0 & \varepsilon_{(01)(10)} & \varepsilon_{(01)(20)} & \varepsilon_{(01)(11)} & \varepsilon_{(01)(22)} & \varepsilon_{(01)(12)} & \varepsilon_{(01)(21)} \\
0 & 0 & \varepsilon_{(02)(10)} & \varepsilon_{(02)(20)} & \varepsilon_{(02)(11)} & \varepsilon_{(02)(22)} & \varepsilon_{(02)(12)} & \varepsilon_{(02)(21)} \\
\varepsilon_{(01)(10)} & \varepsilon_{(02)(10)} & 0 & 0 & \varepsilon_{(10)(11)} & \varepsilon_{(10)(22)} & \varepsilon_{(10)(12)} & \varepsilon_{(10)(21)} \\
\varepsilon_{(01)(20)} & \varepsilon_{(02)(20)} & 0 & 0 & \varepsilon_{(20)(11)} & \varepsilon_{(20)(22)} & \varepsilon_{(20)(12)} & \varepsilon_{(20)(21)} \\
\varepsilon_{(01)(11)} & \varepsilon_{(02)(11)} & \varepsilon_{(10)(11)} & \varepsilon_{(20)(11)} & 0 & 0 & \varepsilon_{(11)(12)} & \varepsilon_{(11)(21)} \\
\varepsilon_{(01)(22)} & \varepsilon_{(02)(22)} & \varepsilon_{(10)(22)} & \varepsilon_{(20)(22)} & 0 & 0 & \varepsilon_{(22)(12)} & \varepsilon_{(22)(21)} \\
\varepsilon_{(01)(12)} & \varepsilon_{(02)(12)} & \varepsilon_{(10)(12)} & \varepsilon_{(20)(12)} & \varepsilon_{(11)(12)} & \varepsilon_{(22)(12)} & 0 & 0 \\
\varepsilon_{(01)(21)} & \varepsilon_{(02)(21)} & \varepsilon_{(10)(21)} & \varepsilon_{(20)(21)} & \varepsilon_{(11)(21)} & \varepsilon_{(22)(21)} & 0 & 0
\end{pmatrix}.$$

Contraction equations $e((ij)\,(kl)\,(mn)) \in \mathcal{S}_3$ should hold for all possible triples of indices (ij), (kl), (mn). It is clear that for a triple where two indices are identical, the equation is automatically fulfilled; the equation also does not depend on the ordering of the triples. The number of equations is then equal to the combination number $\binom{8}{3} = 56$. It is easy to see that equations for which $i+k+m = 0 \wedge j+l+n = 0$ holds will be fulfilled identically and this situation arises in eight cases. Hence the contraction system consists of 48 equations. The matrix group of symmetry H_3 has 48 elements and there exist exactly *two* 24–point orbits of triples of grading indices. We can choose the triples (01)(02)(10) and (01)(10)(11) as representative elements of these orbits. Moreover, all 24 elements from each orbit can be obtained by the action of 24 elements from $SL(2,\mathcal{Z}_3) \subset H_3$ starting from an arbitrary point. Then for our choice of representative points, our system $\mathcal{S}_3$ can be written elegantly as

$$\mathcal{S}_3^a : \varepsilon_{(02)(10)A}\varepsilon_{(01)(12)A} - \varepsilon_{(01)(10)A}\varepsilon_{(02)(11)A} = 0 \tag{35}$$

$$\mathcal{S}_3^b : \varepsilon_{(10)(11)A}\varepsilon_{(01)(21)A} - \varepsilon_{(01)(11)A}\varepsilon_{(10)(12)A} = 0 \quad \forall A \in SL(2,\mathcal{Z}_3)\,, \tag{36}$$

where we used abbreviation $\varepsilon_{(ij)(kl)A} := \varepsilon_{(ij)A,(kl)A}$.

Note that there exist dependent equations; equation obtained from (35) by identity matrix can be written as

$$\varepsilon_{(01)(10)X}\varepsilon_{(02)(11)X} - \varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = 0, \quad X = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{37}$$

This is caused by the fact that [(02)(10)][(01)(12)] and [(01)(10)][(02)(11)] lie in the same orbit with respect to the action of $SL(2,\mathcal{Z}_3)$, where the pairs of indices in bracket [,] and the pairs of these brackets are unordered. The equation generated from equation (37) by the matrix $A = X$

$$\varepsilon_{(01)(10)X^2}\varepsilon_{(02)(11)X^2} - \varepsilon_{(01)(10)X}\varepsilon_{(02)(11)X} = 0 \tag{38}$$

is also contained in $\mathcal{S}_3^a$, due to (35). Adding equations (37) and (38) we have

$$\varepsilon_{(01)(10)X^2}\varepsilon_{(02)(11)X^2} - \varepsilon_{(01)(10)}\varepsilon_{(02)(11)} = 0\,. \tag{39}$$

Since $X^3 = 1$ holds, equation (39) is generated from equation (35) by matrix $A = X^2$. Therefore we can conclude that the left cosets of the group $SL(2,\mathcal{Z}_3)$ with respect to the cyclic subgroup $\{1, X, X^2\}$ then generate just the triples of dependent equations. The number of these cosets according to the Lagrange's theorem is $24/3 = 8$. So we obtained 8 equations (one from each coset) which we eliminate from the system $\mathcal{S}_3^a$. On the other hand, we observe in $\mathcal{S}_3^b$ that the quadruples of indices [(10)(11)][(01)(21)] and [(01)(11)][(10)(12)] do *not* lie in the same orbit and in this way the equations are independent.

4.1 Solution of the contraction system $\mathcal{S}_3$

The goal of this section is to present an algorithm which allows us to determine all equivalence classes (in the sense of Sect. 3.4) of solutions of the nonlinear contraction system.

Theorem 4.1. Let $\mathcal{R}(\mathcal{S})$ be the set of solutions and $\mathcal{I}$ the set of relevant pairs of unordered indices of the contraction system $\mathcal{S}$ of a graded Lie algebra $\Gamma : \mathcal{L} = \bigoplus_{i\in I} \mathcal{L}_i$. For any subsets $\mathcal{Q} \subset \mathcal{R}(\mathcal{S})$ and $\mathcal{P} = \{k_1, k_2, \ldots, k_m\} \subset \mathcal{I}$ we denote

$$\mathcal{R}_0 := \left\{\varepsilon \in \mathcal{R}(\mathcal{S}) \,\middle|\, (\forall \varepsilon' \in \mathcal{Q})(\varepsilon \nsim \varepsilon')\right\},$$
$$\mathcal{R}_1 := \left\{\varepsilon \in \mathcal{R}_0 \,\middle|\, (\forall k \in \mathcal{P})(\varepsilon_k \neq 0)\right\}.$$

Then the solution $\varepsilon \in \mathcal{R}_0$ is inequivalent to all solutions in $\mathcal{R}_1$ *if and only if*

$$\begin{gathered} \varepsilon_{\pi_1(k_1)}\varepsilon_{\pi_1(k_2)}\cdots\varepsilon_{\pi_1(k_m)} = 0 \\ \vdots \\ \varepsilon_{\pi_n(k_1)}\varepsilon_{\pi_n(k_2)}\cdots\varepsilon_{\pi_n(k_m)} = 0 \end{gathered} \tag{40}$$

holds, where $\{\pi_1, \pi_2, \dots, \pi_n\} = \Delta_\Gamma \operatorname{Aut}\Gamma$ is the symmetry group of the grading Γ.

Proof. For any $\varepsilon \in \mathcal{R}_0$ we have (see 31):

$$(\exists \varepsilon' \in \mathcal{R}_1)(\varepsilon \sim \varepsilon') \Leftrightarrow (\exists \varepsilon' \in \mathcal{R}_1)(\exists \alpha)(\exists \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma)(\alpha \bullet \varepsilon^\pi = \varepsilon') \tag{41}$$

$$\Leftrightarrow (\exists \alpha)(\exists \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma)(\alpha \bullet \varepsilon^\pi \in \mathcal{R}_0 \wedge (\alpha \bullet \varepsilon^\pi)_k \neq 0,\ \forall k \in \mathcal{P}) \tag{42}$$

$$\Leftrightarrow (\exists \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma)(\forall k \in \mathcal{P})((\varepsilon^\pi)_k \neq 0)\,. \tag{43}$$

The equivalence (41) is direct consequence of the definition (31), the equivalence (42) expresses the trivial fact that $(\exists \varepsilon' \in \mathcal{R}_1)(\alpha \bullet \varepsilon^\pi = \varepsilon') \Leftrightarrow (\alpha \bullet \varepsilon^\pi \in \mathcal{R}_1)$. Since $\alpha \bullet \varepsilon^\pi \in \mathcal{R}_0$ is for any $\varepsilon \in \mathcal{R}_0$ automatically fulfilled and $(\alpha \bullet \varepsilon^\pi)_k \neq 0 \Leftrightarrow (\varepsilon^\pi)_k \neq 0$, the equivalence (43) follows.

Negating (43) we obtain

$$(\forall \varepsilon' \in \mathcal{R}_1)(\varepsilon \nsim \varepsilon') \Leftrightarrow (\forall \pi \in \Delta_\Gamma \operatorname{Aut}\Gamma)(\exists k \in \mathcal{P})((\varepsilon^\pi)_k = 0)$$

and this is the statement of the theorem. □

We call the system of equations (40) corresponding to the sets $\mathcal{Q} \subset \mathcal{R}(\mathcal{S})$ and $\mathcal{P} \subset \mathcal{I}$ a **non-equivalence system**.

Repeated use of the theorem leads us to the following algorithm for the evaluation of solutions:

1. we set $\mathcal{Q} = \emptyset$ and suppose we have a set of assumptions $\mathcal{P}^0 \subset \mathcal{I}$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S})$, and we explicitly evaluate
$$\mathcal{R}^0 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S}) \mid (\forall k \in \mathcal{P}^0)(\varepsilon_k \neq 0)\right\}$$
and write the *non-equivalence system* $\mathcal{S}^0$ of equations (40) corresponding to $\mathcal{Q} = \emptyset$, $\mathcal{P}^0$.

2. we set $\mathcal{Q} = \mathcal{R}^0$ and suppose we have the set $\mathcal{P}^1 \subset \mathcal{I}$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0)$, and we explicitly evaluate
$$\mathcal{R}^1 = \left\{\varepsilon \in \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0) \mid (\forall k \in \mathcal{P}^1)(\varepsilon_k \neq 0)\right\}$$
and write the non-equivalence system $\mathcal{S}^1$ corresponding to $\mathcal{Q} = \mathcal{R}^0$, $\mathcal{P}^1$.

3. we set $\mathcal{Q} = \mathcal{R}^0 \cup \mathcal{R}^1$. Then $\mathcal{R}_0 = \mathcal{R}(\mathcal{S} \cup \mathcal{S}^0 \cup \mathcal{S}^1)$ and we continue till we have evaluated the whole $\mathcal{R}(\mathcal{S})$ up to equivalence, i.e. till we have arrived at such $\mathcal{Q}$ that the corresponding set $\mathcal{R}_0$ is empty or trivial.

Using the symmetry group of the Pauli grading of $sl(3, \mathbb{C})$, we have evaluated the set of all solutions of the corresponding contraction system up to equivalence. The system of contraction equations $\mathcal{S}_3$ has 2 trivial solutions and 178 non-trivial and non-equivalent ones; of these, 2 solutions depend on two non-zero parameters and further 11 solutions depend on one non-zero parameter. The complete list of solutions will be published elsewhere. It serves as an input to a further analysis — the identification of resulting Lie algebras.

5 Resulting Lie algebras

Structure constants $c_{ij}^k \in \mathbb{C}$ in a basis $\{e_i\}_{i=1}^n$ are defined as usual

$$[e_i, e_j] = c_{ij}^k e_k \, .$$

If we denote by $e_1, \ldots, e_8$ the matrices in (9), i.e.

$$\begin{aligned} e_1 &= Q_3\,, & e_2 &= Q_3^2\,, & e_3 &= P_3\,, & e_4 &= P_3^2\,, \\ e_5 &= P_3Q_3\,, & e_6 &= P_3^2Q_3^2\,, & e_7 &= P_3Q_3^2\,, & e_8 &= P_3^2Q_3\,, \end{aligned}$$

then c_{ij}^k are structure constants of $sl(3,\mathbb{C})$ in the basis of the Pauli grading. Thus the structure constants x_{ij}^k of a contracted Lie algebra $\mathcal{L}^\varepsilon$ corresponding to a solution ε are given by

$$x_{ij}^k = \varepsilon_{ij} c_{ij}^k \, .$$

The procedure for identification of contracted Lie algebras is taken from [16] and we will demonstrate this procedure on a concrete example. Let us consider an example of the solution of the system $\mathcal{S}_3$

$$\varepsilon = \begin{pmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&1&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix} . \tag{44}$$

We denote by $\mathcal{L} = \mathcal{L}^\varepsilon$ corresponding graded contraction. Non-zero commutation relations are

$$\begin{aligned} [e_1, e_3] &= (\omega - 1)e_5\,, & [e_2, e_4] &= (\omega - 1)e_6\,, \\ [e_3, e_5] &= (1 - \omega)e_8\,, & [e_3, e_8] &= (1 - \omega)e_1\,. \end{aligned}$$

At first we compute a **center**

$$C(\mathcal{L}) = \{x \in \mathcal{L} \mid \forall y \in \mathcal{L}\,,\ [x, y] = 0\}$$

and a **derived algebra**

$$D(\mathcal{L}) = [\mathcal{L}, \mathcal{L}]$$

of $\mathcal{L}^\varepsilon$. The result is

$$C(\mathcal{L}) = \{e_6, e_7\}\,, \quad D(\mathcal{L}) = \{e_5, e_6, e_8, e_1\}\,. \tag{45}$$

If the complement of the derived algebra in the center $X = C(\mathcal{L})\backslash D(\mathcal{L})$ is non-empty, then the decomposition of $\mathcal{L}$ can be obtained from the decomposition of the quotient algebra

$$\mathcal{L}/D(\mathcal{L}) = X/D(\mathcal{L}) \oplus \tilde{\mathcal{L}}/D(\mathcal{L})\,,$$

where $D(\mathcal{L}) \subset \tilde{\mathcal{L}}$. In our case the complement of the derived algebra is $\{e_7\}$ and Lie algebra $\mathcal{L}$ is direct sum as follows

$$\mathcal{L} = \{e_7\} \oplus \{e_5, e_6, e_8, e_1, e_2, e_3, e_4\} . \tag{46}$$

After the separation of the central component, we denote remaining 7–dimensional algebra by the same symbol $\mathcal{L}$. Dimensionality of a **centralizer of the adjoint representation** in the ring $R = \mathbb{C}^{n,n}$

$$C_R(\mathrm{ad}(\mathcal{L})) = \{x \in R \mid \forall y \in \mathrm{ad}(\mathcal{L})\,,\ [x, y] = 0\}$$

is in our case $\dim(C_R(\mathrm{ad}(\mathcal{L}))) = 5$. Lie algebra $\mathcal{L}$ is decomposable into a direct sum of its ideals if and only if there exists a non-trivial **idempotent** in $C_R(\mathrm{ad}(\mathcal{L}))$, i.e. its element with the propriety

$$0 \neq E \neq 1\,, \quad E^2 = E\,.$$

In such a case the decomposition has the form

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1\,, \quad [\mathcal{L}_0, \mathcal{L}_1] = 0\,, \quad [\mathcal{L}_i, \mathcal{L}_i] \subseteq \mathcal{L}_i\,, \quad i = 0, 1\,,$$

where $\mathcal{L}_0, \mathcal{L}_1$ are eigen–subspaces of the idempotent E corresponding to the eigenvalues $0, 1$. A matrix

$$M = \mathrm{diag}\,(0, 1, 0, 0, 1, 0, 1) \tag{47}$$

is a non-trivial idempotent in $C_R(\mathrm{ad}(\mathcal{L}))$. Eigen–subspaces of M are subalgebras of $\mathcal{L}$:

$$\mathcal{L}^0 = \{e_5, e_8, e_1, e_3\}\,, \quad \mathcal{L}^1 = \{e_6, e_2, e_4\}\,, \tag{48}$$

Algebra $\mathcal{L}$ is thus the direct sum of its subalgebras

$$\mathcal{L} = \{e_5, e_8, e_1, e_3\} \oplus \{e_6, e_2, e_4\} = \mathcal{L}^0 \oplus \mathcal{L}^1\,. \tag{49}$$

Subalgebras $\mathcal{L}^i$, $i = 0, 1$ are further indecomposable. We compute a **derived sequence**

$$D^0(\mathcal{L}) \supseteq D^1(\mathcal{L}) \supseteq \ldots \supseteq D^k(\mathcal{L}) \supseteq \ldots$$
$$D^0(\mathcal{L}) = \mathcal{L}\,, \qquad D^{k+1}(\mathcal{L}) = [D^k(\mathcal{L}), D^k(\mathcal{L})]$$

for both algebras

$$\begin{array}{llll} D^0(\mathcal{L}^0) = \{e_5, e_8, e_1, e_3\}\,, & D^1(\mathcal{L}^0) = \{e_5, e_8, e_1\}\,, & D^k(\mathcal{L}^0) = \{0\}\,, & k \geq 2\,, \\ D^0(\mathcal{L}^1) = \{e_6, e_2, e_4\}\,, & D^1(\mathcal{L}^1) = \{e_6\}\,, & D^k(\mathcal{L}^1) = \{0\}\,, & k \geq 2\,. \end{array} \tag{50}$$

and we conclude that algebras $\mathcal{L}^0$, $\mathcal{L}^1$ are *solvable*. Afterwards we evaluate a **lower central sequence**

$$(\mathcal{L})^0 \supseteq (\mathcal{L})^1 \supseteq \ldots \supseteq (\mathcal{L})^k \supseteq \ldots$$
$$(\mathcal{L})^0 = \mathcal{L}\,, \qquad (\mathcal{L})^{k+1} = [(\mathcal{L})^k, \mathcal{L}]$$

for these algebras

$$\begin{aligned} &(\mathcal{L}^0)^0 = \{e_5, e_8, e_1, e_3\}\,, \quad (\mathcal{L}^0)^k = \{e_5, e_8, e_1\}\,, \quad k \geq 1\,, \\ &(\mathcal{L}^1)^0 = \{e_6, e_2, e_4\}\,, \quad (\mathcal{L}^1)^1 = \{e_6\}\,, \quad (\mathcal{L}^1)^k = \{0\}\,, \quad k \geq 2\,. \end{aligned} \tag{51}$$

Hence we see that $\mathcal{L}^1$ is moreover *nilpotent*. Finally an **upper central sequence**

$$C^0(\mathcal{L}) \subseteq C^1(\mathcal{L}) \subseteq \ldots \subseteq C^k(\mathcal{L}) \subseteq \ldots$$
$$C^0(\mathcal{L}) = 0\,, \qquad C^{k+1}(\mathcal{L})/C^k(\mathcal{L}) = C(\mathcal{L}/C^k(\mathcal{L}))$$

is for both algebras evaluated as follows

$$\begin{aligned} &C^k(\mathcal{L}^0) = \{0\}\,, \quad k \geq 0\,, \\ &C^0(\mathcal{L}^1) = \{0\}\,, \quad C^1(\mathcal{L}^1) = \{e_6\}\,, \quad C^k(\mathcal{L}^1) = \{e_6, e_2, e_4\}\,, \quad k \geq 2\,. \end{aligned} \tag{52}$$

A **nilradical** is a maximal nilpotent ideal and algebra $\mathcal{L}^0$ has 3–dimensional abelian nilradical $\{e_5, e_8, e_1\}$. It turned out that all graded contractions of the Pauli graded $sl(3,\mathbb{C})$ are solvable or nilpotent algebras and thus the **Levi decomposition** into a semidirect sum of a radical $R(\mathcal{L})$ and a semi-simple subalgebra are, like in this case, trivial.

Computing an algebra of derivations is not a part of the identification procedure in [16], but we found its invariant dimension very useful in determining the classes of non-isomorphic results. An **algebra of derivations** $\mathcal{D}(\mathcal{L}) \subset gl(\mathcal{L})$ contains such linear mappings D, which for all $x, y \in \mathcal{L}$ satisfy

$$D\,[x,y] = [Dx, y] + [x, Dy]\,,$$

and in our case

$$\dim \mathcal{D}(\mathcal{L}^0) = \dim \mathcal{D}(\mathcal{L}^1) = 6\,.$$

We now finish the identification by comparing with names of algebras in [13] and we have

$$\mathcal{L} = \{e_7\} \oplus \{e_5, e_8, e_1, e_3\} \oplus \{e_6, e_2, e_4\} = \{e_7\} \oplus A_{4,6}\left(\frac{\pm 2}{\sqrt{3}}, \frac{\mp 1}{\sqrt{3}}\right) \oplus A_{3,1}\,. \tag{53}$$

In the Table 1 are summarized numbers of results of graded contractions of the Pauli graded $sl(3,\mathbb{C})$ after the identification process similar to presented example. In the first column are the dimensions of non-Abelian parts of Lie algebras and in other columns are the numbers of obtained Lie algebras. The complete tables with results, as well as more detailed identification, will be published elsewhere.

Together with the 8–dimensional Abelian Lie algebra and the original Lie algebra $sl(3,\mathbb{C})$ is the number of all contracted algebras 141.

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Table 1. Summary of graded contractions of the Pauli graded $sl(3,\mathbb{C})$

Dimension	Solvable		Nilpotent		Summary
	Indecom.	Decom.	Indecom.	Decom.	
3			1		1
4	1		1		2
5	1		4		5
6	1		8	1	10
7	4	1	24	1	30
8	11	2	75	3	91

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