Some features about toric quaternionic Kahler geometry in $D = 4$

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We explain some features about toric self dual structures and toric quaternionic Kahler manifolds in four dimensions. Applications are outlined.

1   Self-dual structures with two commuting isometries

Let us recall that for an Euclidean space in $D = 4$ the rotation group $SO(4)$ is locally isomorphic to $SU(2) \times SU(2)$ and therefore the Weyl tensor W decomposes as $W = W_+ \oplus W_-$ where the components $W_\pm$ corresponds to one of the $SU(2)$ groups. W is by definition the conformal invariant part of the Riemann tensor, this means that is unchanged under an scaling $g \to \Omega^2 g$. A conformal structure $[g]$ is defined as the family of metrics obtained from g by conformal transformations. If $W_- = 0$ for a given g of $[g]$ then g is called self-dual and, by conformal invariance, $[g]$ will be a self-dual structure.

Let us focus in families $[g]$ with two commuting $U(1)$ isometries. Then the representatives g of $[g]$ takes locally the Gowdy form

$$g = g_{ab}\mathrm{d}x^a \mathrm{d}x^b + g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \,. \tag{1}$$

The latin indices a, b corresponds to vectors on N and the greek indices α, β to vectors on T^2. Both g_{ab} and $g_{\alpha\beta}$ are supposed to be independent of $x^\alpha = (\theta, \varphi)$. It is seen that the Killing vectors are $\partial/\partial\theta$ and $\partial/\partial\varphi$ and the level surfaces of constant θ and φ are orthogonal to both Killing fields.

By a theorem due to Gauss there exists a local scale transformation $g \to \Omega^2 g$ which reduce (1) to

$$g = \mathrm{d}\rho^2 + \mathrm{d}\eta^2 + \widetilde{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \,. \tag{2}$$

Because self-duality is a property of $[g]$ rather than g there is not loss of generality in consider the anzatz (2) instead of (1). Define the basis (e_1, e_2) such that

$$\widetilde{g}_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = e_1^2 + e_2^2 \,.$$

1

There exist a linear transformation T connecting the basis $(\rho d\theta, \rho d\varphi)$ with (e_1, e_2), which we will write as

$$T = \begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix},$$

where A_i and B_i are certain functions of (ρ, η). By calculating T^{-1} it is seen that the angular part of g can be expressed as

$$\widetilde{g}_{\alpha\beta}dx^\alpha dx^\beta = \frac{(\rho A_0 d\theta - \rho B_0 d\varphi)^2 + (\rho A_1 d\theta - \rho B_1 d\varphi)^2}{(A_0 B_1 - A_1 B_0)^2}. \tag{3}$$

The advantage of this form is that the self-duality condition is equivalent to a system of linear equations. Imposing the condition $W_- = 0$ for (2) gives the following proposition [1]:

Proposition 1 *Any self-dual g with two commuting killing vectors $\partial/\partial\theta$ and $\partial/\partial\varphi$ over $M = N \times T^2$ is locally conformal to a self-dual metric g_j of the form*

$$g_j = (A_0 B_1 - A_1 B_0)\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{(A_0 d\theta - B_0 d\varphi)^2 + (A_1 d\theta - B_1 d\varphi)^2}{A_0 B_1 - A_1 B_0}, \tag{4}$$

where the functions A^i satisfies

$$(A_0)_\rho + (A_1)_\eta = \frac{A_0}{\rho}, \tag{5}$$

$$(A_0)_\eta - (A_1)_\rho = 0, \tag{6}$$

and the same equations holds for B_i.

Equations (5) and (6) are equivalent to the condition $W_- = 0$. The Joyce metrics (4) are obtained by introducing (3) in (2) and making a conformal scaling with a factor $(A_0 B_1 - A_1 B_0)/\rho^2$. Such form is more convenient in order to find the Einstein metrics among the Joyce ones. Therefore the problem to find toric self-dual structures in $D = 4$ has been reduced to solve a linear system for A^i and independently for B^i. The original proof of proposition 1 has been obtained in a rather different way than here; it is based on a method discussed in Appendix B.

It should be noted that (6) implies that

$$A_0 = G_\rho, \quad A_1 = G_\eta \tag{7}$$

for certain potential function G. Then (5) implies that $G_{\rho\rho} + G_{\eta\eta} = G_\rho/\rho$. Conversely (5) implies that

$$A_0 = -\rho V_\eta, \quad A_1 = \rho V_\rho \tag{8}$$

and (6) gives the Ward monopole equation [5]

$$V_{\eta\eta} + \rho^{-1}(\rho V_\rho)_\rho = 0, \tag{9}$$

2

which has been proven to describe hyperkahler metrics with two commuting isometries. The relations

$$G_\rho = -\rho V_\eta = A_0 \,, \quad G_\eta = \rho V_\rho = A_1 \tag{10}$$

constitute a Backlund transformation allowing to find a monopole V starting with a known G or viceversa. The functions B_i can be also expressed in terms of another potential functions G' and V' satisfying the same equations than V and G.

2 The Toda structure

In the first section we have described the families of self-dual structures with T^2 isometry. In this section we extract the Einstein representatives among the Joyce ones, in four dimensions an Einstein self-dual metric is automatically quaternionic Kahler. In order to achieve this we should use the following theorem [3]:

Proposition 2 *Any self-dual Einstein metric g with one Killing vector in $D = 4$ there exist a system of coordinates (x, y, z, t) for which takes the form*

$$g = \frac{1}{z^2} \left[U\left(e^u(dx^2 + dy^2) + dz^2\right) + \frac{1}{U}(dt + A)^2 \right]. \tag{11}$$

The functions (U, A, u) are independent of the variable t and satisfies

$$(e^u)_{zz} + u_{yy} + u_{xx} = 0 \,, \tag{12}$$

$$dA = U_x dz \wedge dy + U_y dx \wedge dz + (Ue^u)_z dy \wedge dx \,, \tag{13}$$

$$U = 2 - zu_z \,. \tag{14}$$

Conversely, any solution of (12), (12) and (14) define by (11) a self-dual Einstein metric.

Equation (14) is the Einstein condition and if it is relaxed we get Jones–Tod correspondence [4]. The integrability condition for the existence of A is

$$(Ue^u)_{zz} + U_{yy} + U_{xx} = 0 \tag{15}$$

and is easily seen that (14) satisfies it. Then the problem to find the Einstein metrics among the Joyce ones is to reduce them to the form (11) and then to apply (14). The result will be an extra relation between the functions A_i and B_i and the resulting metrics will be toric self dual Einstein.

The first task is to find a new coordinate system (x, y, z, t) for the Joyce metrics (4) defined in terms of the old one $(\rho, \eta, \theta, \varphi)$ for which they are expressed as

$$g = U\left(e^u(dx^2 + dy^2) + dz^2\right) + \frac{1}{U}(dt + A)^2 \,. \tag{16}$$

3

according to (11). To do this it is needed to write (4) as

$$g_j = \frac{A_0 B_1 - A_1 B_0}{\rho^2 (A_0^2 + A_1^2)} \left((A_0^2 + A_1^2)(\mathrm{d}\rho^2 + \mathrm{d}\eta^2) + \rho^2 \mathrm{d}\varphi^2 \right) +$$
$$+ \frac{A_0^2 + A_1^2}{A_0 B_1 - A_1 B_0} \left(\mathrm{d}\theta - \frac{(A_0 B_0 + A_1 B_1)\mathrm{d}\varphi}{A_0^2 + A_1^2} \right)^2 \tag{17}$$

and is seen that after rescaling by ρ and identifying $t = \theta$ that it takes the form (11) with

$$\mathrm{e}^u(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2 = (A_0^2 + A_1^2)(\mathrm{d}\rho^2 + \mathrm{d}\eta^2) + \rho^2 \mathrm{d}\varphi^2 \,, \tag{18}$$

$$A = -\frac{(A_0 B_0 + A_1 B_1)}{A_0^2 + A_1^2}\mathrm{d}\varphi \,, \quad U = \frac{A_1 B_0 - A_0 B_1}{\rho(A_0^2 + A_1^2)} \,. \tag{19}$$

The factor ω can be calculated through $\mathrm{d}A = *_h(\mathrm{d}U - U\omega)$ and is

$$\omega = -\frac{2A_0}{\rho(A_0^2 + A_1^2)}\,\mathrm{d}G \,, \quad \mathrm{d}G = -\rho V_\eta \mathrm{d}\rho + \rho V_\rho \mathrm{d}\eta \,. \tag{20}$$

The next problem to find the coordinates (x, y, z) as functions of (ρ, η, φ) for which (18) holds. Using the equation we can write the two differentials (9)

$$\mathrm{d}V = V_\rho \mathrm{d}\rho + V_\eta \mathrm{d}\eta \,,$$
$$\mathrm{d}G = \rho(V_\rho \mathrm{d}\eta - V_\eta \mathrm{d}\rho)$$

and it can be easily checked that

$$\mathrm{d}G^2 + \rho^2 \mathrm{d}V^2 = \rho^2(V_\eta^2 + V_\rho^2)(\mathrm{d}\rho^2 + \mathrm{d}\eta^2) = (A_0^2 + A_1^2)(\mathrm{d}\rho^2 + \mathrm{d}\eta^2) \,,$$

where in the last step formula (8) has been used. From the last expression is seen that (18) is

$$\mathrm{e}^u(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2 = \rho^2(\mathrm{d}V^2 + \mathrm{d}\varphi^2) + \mathrm{d}G^2 \,. \tag{21}$$

Equation (21) shows that a solution $u(x, z)$ of the continuum Toda equation is defined by the identifications

$$\mathrm{e}^u = \rho^2 \,, \quad x = V \,, \quad y = \varphi \,, \quad z = G \,. \tag{22}$$

The solution u is independent of y is due to the presence of the other isometry, which is also a symmetry of h. Formula (22) defines the coordinate system that we were looking for.

At first sight (22) relates the solutions of the axially symmetric Toda equation with two solutions V and G of two different linear differential equations. But they are related by a Backlund transformation and it can be directly checked that if V is a Ward monopole, then W such that $W_\eta = V$ is also a Ward monopole and it follows that $G = \rho W_\rho$. Inserting the expressions in terms of W in (22) and changing the notation replacing W by V by convenience gives the following proposition [5]:

4

Proposition 3 *Any solution V of the equation $V_{\eta\eta} + \rho^{-1}(\rho V_\rho)_\rho = 0$ defines locally the coordinate system (x, z)*

$$x = V_\eta, \quad z = \rho V_\rho, \tag{23}$$

in terms of (ρ, η) and conversely (23) defines implicitly (ρ, η) as functions of (x, t). Then the function $u(x, z) = \log(\rho^2)$ is a solution of the axially symmetric Toda equation

$$(\mathrm{e}^u)_{zz} + u_{xx} = 0. \tag{24}$$

This procedure can be inverted in order to find a Ward monopole V starting with a given Toda solution.

Proposition 3 gives a method to find solutions of a non linear equation (the continuum Toda one) starting with a solution of a linear one (the Ward equation). But it is difficult in practice to find explicit solutions of (24) and usually proposition 4 gives implicit solutions.

An important detail is that the Toda structure (18) and the Toda solution u are completely determined just in terms of A_i. Only the monopole (U, A, ω) depends on both A_i and B_i, which are not related in any way.

For Joyce spaces the relation $\omega = -u_z \mathrm{d}z$ and (20) gives

$$u_z = \frac{A_0}{\rho(A_0^2 + A_1^2)}, \quad 2 - z u_z = \frac{\rho(A_0^2 + A_1^2) - G A_0}{\rho(A_0^2 + A_1^2)}.$$

Then the insertion of the expression for U (19) in terms of A_i and B_i into the Einstein condition $U = 2 - z u_z$ gives

$$A_1 B_0 - A_0 B_1 = \rho(A_0^2 + A_1^2) - G A_0.$$

Thus $B_0 = \rho A_1 + \xi_0$ and $B_1 = G - \rho A_0 + \xi_1$ with $A_1 \xi_0 = A_0 \xi_1$. The functions ξ_i are determined by asking B_i to satisfy the Joyce system (5) and (6), the result is $\xi_0 = -\eta A_0$ and $\xi_1 = -\eta A_1$. Therefore the metric $g_j/\rho z^2$ is Einstein if and only if

$$A_0 = G_\rho, \qquad A_1 = G_\eta, \tag{25}$$
$$B_0 = \eta G_\rho - \rho G_\eta, \quad B_1 = \rho G_\rho + \eta G_\eta - G, \tag{26}$$

which is the Calderbank–Pedersen solution. Defining $G = \sqrt{\rho} F$ it follows that F satisfies

$$F_{\rho\rho} + F_{\eta\eta} = \frac{3F}{4\rho^2}.$$

Then inserting (25) and (26) expressed in terms of F into $g_j/\rho z^2$ and making the identification $z = G$ gives the following proposition [2]:

Proposition 4 *For any Einstein–metric with self-dual Weyl tensor and nonzero scalar curvature possessing two linearly independent commuting Killing fields there*

5

exists a coordinate system in which the metric g has locally the form

$$\mathrm{d}s^2 = \frac{F^2 - 4\rho^2(F_\rho^2 + F_\eta^2)}{4F^2} \frac{\mathrm{d}\rho^2 + \mathrm{d}\eta^2}{\rho^2} +$$
$$+ \frac{[(F - 2\rho F_\rho)\alpha - 2\rho F_\eta \beta]^2 + [(F + 2\rho F_\rho)\beta - 2\rho F_\eta \alpha]^2}{F^2[F^2 - 4\rho^2(F_\rho^2 + F_\eta^2)]} , \tag{27}$$

where $\alpha = \sqrt{\rho}\mathrm{d}\theta$ and $\beta = \dfrac{\varphi + \eta \mathrm{d}\theta}{\sqrt{\rho}}$ and $F(\rho, \eta)$ is a solution of the equation

$$F_{\rho\rho} + F_{\eta\eta} = \frac{3F}{4\rho^2} . \tag{28}$$

on some open subset of the half–space $\rho > 0$. On the open set defined by $F^2 > 4\rho^2(F_\rho^2 + F_\eta^2)$ the metric g has positive scalar curvature, whereas $F^2 < 4\rho^2(F_\rho^2 + F_\eta^2)$, g is self-dual with negative scalar curvature.

The Einstein condition $R_{ij} = \kappa g_{ij}$ is not invariant under scale transformations, so Proposition 4 gives all the quaternionic–Kahler metrics with T^2 isometry up to a constant multiple. The problem to find them is reduced to find an F satisfying the linear equation (28), that is, an eigenfunction of the hyperbolic laplacian with eigenvalue $3/4$.

To finish we recall that this result has many applications in different areas of physics. For an account references [8] – [16] can be useful.

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