Similarity solutions of an equation describing ice sheet dynamics

R.J. Wiltshire

The Division of Mathematics and Statistics, The University of Glamorgan
Pontypridd CF37 1DL, Great Britain

This paper focus's upon the derivation of the similarity solutions of a free boundary problem arising in glaciology. With reference to shallow ice sheet flow we present a potential symmetry analysis of the second order non-linear degenerate parabolic equation that describe non-Newtonian ice sheet dynamics in the isothermal case. A full classical and also a non-classical symmetry analysis is presented. A particular example of a similarity solution to a problem formulated with Cauchy boundary conditions is described. This demonstrates the existence of a free moving boundary and also an accumulation–ablation function with realistic physical properties.

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1 Introduction

In recent years there has been much interest on modelling ice sheet dynamics especially because of its importance in the understanding of global climate change, global energy balance and circulation models. Although various physical theories for large ice sheet motion have been presented there exists no general mathematical treatment. In this paper we consider an obstacle formulation of slow, isothermal, one dimensional ice slow on a rigid bed due to Fowler [1] .

The model describing the ice sheet dynamics is formulated in terms of the following one dimensional non-linear degenerate diffusion equation:

$$\Psi\left(x,t,u,u_t,u_x,u_{xx}\right) \equiv u_t - a - \left(\frac{u^{n+2}\left|u_x\right|^{n-1}u_x}{n+2} - fu\right)_x = 0\,, \tag{1}$$

where a suffix indicates a partial derivative. Moreover $u = u\left(x,t\right) > 0$ is the top surface of the sheet, $a = a\left(x,t\right)$ is the accumulation–ablation rate function, $f = f\left(x,t\right)$ is the basal velocity, n is the Glen exponent (typically $n \approx 3$). The physical problem may be characterized by the following properties as have recently been discussed by Calvo et al [2]:

1. Given an initial ice sheet initial $u\left(x,0\right)$, and known $a\left(x,t\right)$, $f\left(x,t\right)$ to use the partial differential equation. to find $u\left(x,t\right)$ over a parabolic domain.

2. An ice free region (melt zone) $u\left(x,t\right) = 0$ exists which defines two free boundaries $x_-\left(t\right)$ and $x_+\left(t\right)$.

3. Physically admissible solutions are non-negative $u\left(x,t\right) \geq 0$ for which $a > 0$ except in a region near the two free boundaries where $a < 0$.

It is the aim here to show that similarity solutions compatible with these statements do exist. To assist the mathematical analysis the ice sheet model will be written in a conserved or potential form, namely the first order system, $\mathbf{\Psi} \equiv (\Psi_1, \Psi_2) = \mathbf{0}$ where:

$$\begin{aligned} \Psi_1 &= v_x - u + \lambda = 0\,, \\ \Psi_2 &= v_t - \frac{u^{n+2}\left|u_x\right|^{n-1} u_x}{n+2} + fu = 0\,, \end{aligned} \tag{2}$$

where $v = v\left(x,t\right)$ is the potential function and where $\lambda = \lambda\left(x,t\right)$ is such that

$$a \equiv \lambda_t\,. \tag{3}$$

It is noted that Bluman et al [3], [4] first introduced the method for finding new classes of symmetries when partial differential equations can be written in potential form. They demonstrated that the Lie point symmetries of the potential system induce non-Lie contact symmetries for the original partial differential equation.

2 Potential symmetry analysis of the ice sheet equation

In the classical Lie group method, one–parameter infinitesimal point transformations,with group parameter ε are applied to the dependent and independent variables (x,t,u,v). In this case the transformation, including that of the potential variable are

$$\begin{aligned} \bar{x} &= x + \varepsilon\eta_1\left(x,t,u,v\right) + O\left(\xi^2\right), \qquad & \bar{t} &= t + \varepsilon\eta_2\left(x,t,u,v\right) + O\left(\varepsilon^2\right), \\ \bar{u} &= u + \varepsilon\phi_1\left(x,t,u,v\right) + O\left(\xi^2\right), \qquad & \bar{v} &= v + \varepsilon\phi_2\left(x,t,u,v\right) + O\left(\varepsilon^2\right) \end{aligned} \tag{4}$$

and the Lie method requires form invariance of the solution set:

$$\Sigma \equiv \left\{u(x,t), v(x,t),\ \mathbf{\Psi} = 0\right\}. \tag{5}$$

This results in a system of overdetermined, linear equations for the infinitesimals η_1, η_2, ϕ_1 and ϕ_2. The corresponding Lie algebra of symmetries is the set of vector fields

$$\mathcal{X} = \eta_1\left(x,t,u,v\right)\frac{\partial}{\partial x} + \eta_2\left(x,t,u,v\right)\frac{\partial}{\partial t} + \phi_1\left(x,t,u,v\right)\frac{\partial}{\partial u} + \phi_2\left(x,t,u,v\right)\frac{\partial}{\partial v}\,. \tag{6}$$

The condition for invariance of (1) is the equation

$$\mathcal{X}_E^{(1)}\left(\mathbf{\Psi}\right)|_{\Psi_1=0,\Psi_2=0} = 0\,, \tag{7}$$

where the first prolongation operator $\mathcal{X}_E^{(1)}$ is written in the form

$$\mathcal{X}_E^{(2)} = \mathcal{X} + \phi_1^{[t]}\frac{\partial}{\partial u_t} + \phi_1^{[x]}\frac{\partial}{\partial u_x} + \phi_2^{[t]}\frac{\partial}{\partial v_t} + \phi_2^{[x]}\frac{\partial}{\partial v_x}\,, \tag{8}$$

where $\phi_1^{[t]}$, $\phi_1^{[x]}$ and $\phi_2^{[t]}$, $\phi_2^{[x]}$ are defined through the transformations of the partial derivatives of u and v. In particular:

$$\begin{aligned}
\bar{u}_{\bar{x}} &= u_x + \varepsilon\phi_1^{[x]}(x,t,u,v) + O\left(\varepsilon^2\right), \\
\bar{u}_{\bar{t}} &= u_t + \varepsilon\phi_1^{[t]}(x,t,u,v) + O\left(\varepsilon^2\right), \\
\bar{v}_{\bar{x}} &= v_x + \varepsilon\phi_2^{[x]}(x,t,u,v) + O\left(\varepsilon^2\right), \\
\bar{v}_{\bar{t}} &= v_t + \varepsilon\phi_2^{[t]}(x,t,u,v) + O\left(\varepsilon^2\right).
\end{aligned} \tag{9}$$

Once the infinitesimals are determined the symmetry variables may be found from condition for invariance of surfaces $u{=}u\,(x,t)$ and $v{=}v\,(x,t)$:

$$\begin{aligned}
\Omega_1 &= \phi_1 - \eta_1 u_x - \eta_2 u_t = 0\,, \\
\Omega_2 &= \phi_2 - \eta_1 v_x - \eta_2 v_t = 0\,.
\end{aligned} \tag{10}$$

In the following the Macsyma program symmgrp.max [5] and Maple software have been used to calculate the determining equations. In the case of the ice equation (2) there are nine over-determined linear determining equations. From these equations it may be shown that:

$$\eta_1 = \eta_1(x,t) = (c_0 - z)\,x + s\,, \tag{11}$$

$$\eta_2 = \eta_2(t)\,, \tag{12}$$

$$\phi_1 = \phi_1(t,u) = zu\,, \tag{13}$$

$$\phi_2 = \phi_2(x,t,v) = g + c_0 v\,, \tag{14}$$

where $s = s\,(t)$, $z = z\,(t)$ and $g = g\,(x,t)$ and c_0 is an arbitrary constant such that:

$$(3n+2)\,z + \eta_{2_t} - (n+1)c_0 = 0\,, \tag{15}$$

$$(zx - s - c_0 x)\,\lambda_x - \eta_2\lambda_t + z\lambda - g_x = 0\,, \tag{16}$$

$$x\lambda z_t - \lambda s_t - g_t = 0\,, \tag{17}$$

$$(zx - s - c_0 x)\,f_x - \eta_2 f_t = -f\big((3n+1)z - nc_0\big) + xz_t - s_t\,. \tag{18}$$

When it is assumed that $s\,(t)$ and $z\,(t)$ are known then equation (15) may be used to determine $\eta_2\,(t)$ whilst (16) to (18) may be used to determine $g\,(x,t)$, $\lambda\,(x,t)$ and $f\,(x,t)$.

We observe that we have shown that the potential symmetries of the conserved form of the ice dynamics equations (2) are entirely equivalent to those of single equation.(1). This is so because according to [4] additional symmetries can only be induced by the potential system when:

$$\eta_{1_v}^2 + \eta_{2_v}^2 + \phi_{1_v}^2 \neq 0\,. \tag{19}$$

Clearly substitution of equations (11), (12) and (13) demonstrate that this is not the case.

In addition that a differential consequence of equations (16) and (17) incorporating the relation (3) is the differential equation for a, similar in form to (18), namely:

$$(zx - s - c_0 x)\, a_x - \eta_2 a_t = -a(n+1)\,(3z - c_0)\,. \tag{20}$$

Moreover, we note that equation (16) may be obtained directly by differentiating the second surface invariant condition (10) with respect to x and then applying (2), (11) – (14) together with the first of (10).

In summary, the results (15) to (20) together with the first invariant condition at (10) may be simplified by eliminating z using (15) and combined to give three first order partial differential equations which $u\,(x,t)$, $a\,(x,t)$ and $f\,(x,t)$ must satisfy, namely:

$$\left(s + \frac{((2n+1)\,c_0 + r_t)}{3n+2}x\right) u_x + r u_t = \frac{(n+1)\,c_0 - r_t}{3n+2}u\,, \tag{21}$$

$$\left(s + \frac{((2n+1)\,c_0 + r_t)}{3n+2}x\right) a_x + r a_t = \frac{(n+1)}{3n+2}\,(c_0 - 3r_t)\,a\,, \tag{22}$$

$$\left(s + \frac{((2n+1)\,c_0 + r_t)}{3n+2}x\right) f_x + r f_t = \frac{((2n+1)\,c_0 - (3n+1)\,r_t)}{3n+2}f + \frac{x r_{tt}}{3n+2} + s_t\,, \tag{23}$$

where $r\,(t) \equiv \eta_2\,(t)$ has been used to simplify the notation.

3 Solutions for the case $n = 3$

The exponent n which occurs in (1) is the so called Glen's exponent and Fowler [1] suggests that $n \approx 3$ in physically realistic situations Thus in the following we will assume that $n = 3$ although the analysis is unchanged for any non-Newtonian values $n > 1$. The results presented assumed that each of the functions u, a and f explicitly depend on x and t.

3.1 The case $f = 0$

Firstly, substitution of $f = 0$ into equation (23) gives $r(t) = c_1 t + c_2$ and $s(t) = c_3$.

3.1.1 The subcase $7c_0 + c_1 \neq 0$, $c_1 \neq 0$

The solution of (21) and (22) may be expressed in terms of the similarity variable $\omega = \omega\,(x,t)$ for which:

$$\omega\,(x,t) = (x + c_3)\,(c_1 t + c_2)^{-(7c_0+c_1)/(11c_1)}\,, \quad \text{when } 7c_0 + c_1 \neq 0\,, \tag{24}$$

with:

$$u\,(x,t) = \psi\,(\omega)\,(c_1 t + c_2)^{(4c_0-c_1)/(11c_1)}\,, \tag{25}$$

$$a\,(x,t) = A\,(\omega)\,(c_1 t + c_2)^{(4c_0-12c_1)/(11c_1)}\,. \tag{26}$$

Substituting the relationships into equation (1) with $n = 3$ gives rise to the ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\psi^5\psi_\omega^3}{5} + \frac{(c_1 + 7c_0)\omega\psi}{11}\right) - c_0\psi - A = 0\,. \tag{27}$$

3.1.2 The subcase $7c_0 + c_1 = 0$, $c_1 \neq 0$

For this subcase it may be shown that:

$$\omega\,(x,t) = x + c_3 \ln\left(c_1 t + c_2\right), \quad \text{when } \; 7c_0 + c_1 = 0,, \tag{28}$$

with

$$u\,(x,t) = \psi\,(\omega)\,(c_1 t + c_2)^{-1/7}\ , \tag{29}$$

$$a\,(x,t) = A\,(\omega)\,(c_1 t + c_2)^{-8/7}\ . \tag{30}$$

Substituting the relationships into equation (1) with $n = 3$ gives rise to the ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\psi^5\psi_\omega^3}{5} + 7c_0c_3\psi\right) - c_0\psi - A = 0\,. \tag{31}$$

3.1.3 The subcase $c_1 = 0$

Without loss of generality consider the case $c_2 = 1$. The solution of (21) and (22) may be expressed in terms of the similarity variable $\omega = \omega\,(x,t)$ for which:

$$\omega\,(x,t) = (x + c_3)\,\mathrm{e}^{-(7c_0t)/(11)}\,, \tag{32}$$

with:

$$u\,(x,t) = \psi\,(\omega)\,\mathrm{e}^{4c_0t/11}\,, \tag{33}$$

$$a\,(x,t) = A\,(\omega)\,\mathrm{e}^{4c_0t/11}\,. \tag{34}$$

Substituting the relationships into equation (1) with $n = 3$ gives rise to the ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\psi^5\psi_\omega^3}{5} + \frac{7c_0\omega\psi}{11}\right) - c_0\psi - A = 0\,. \tag{35}$$

3.2 The case $s = 0$, $r \neq 0$, $f \neq 0$

In this case equations (21) to (23) may be integrated immediately to give solutions in terms of the similarity variable $\omega = \omega\,(x,t)$ for which:

$$\omega\,(x,t) = xr^{-1/11}\exp\left(-\frac{7c_0}{11}\int\frac{\mathrm{d}t}{r}\right), \tag{36}$$

with:

$$u\,(x,t) = \psi\,(\omega)\,r^{-1/11}\exp\left(\frac{4c_0}{11}\int\frac{\mathrm{d}t}{r}\right), \tag{37}$$

$$a\,(x,t) = A\,(\omega)\,r^{-12/11}\exp\left(\frac{4c_0}{11}\int\frac{\mathrm{d}t}{r}\right), \tag{38}$$

$$f\,(x,t) = \left(\frac{\omega r_t}{11} + F(\omega)\right)r^{-10/11}\exp\left(\frac{7c_0}{11}\int\frac{\mathrm{d}t}{r}\right). \tag{39}$$

Substituting the relationships into equation (1) with $n = 3$ gives rise to the ordinary differential equation:

$$\tfrac{3}{5}\,\psi^5\psi_\omega^2\psi_{\omega\omega} + \psi^4\psi_\omega^4 + \tfrac{7}{11}\,c_0\omega\psi_\omega - \tfrac{4}{11}\,c_0\psi - \psi F_\omega - \psi_\omega F - A = 0\,. \tag{40}$$

That is:

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\psi^5\psi_\omega^3}{5} + \frac{7c_0\omega\psi}{11} - \psi F\right) - c_0\psi - A = 0\,. \tag{41}$$

3.3 The case $s \neq 0$, $r \neq 0$, $f \neq 0$

In this case the similarity variable has the form:

$$\omega\,(x,t) = xr^{-1/11}\exp\left(-\frac{7c_0}{11}\int\frac{\mathrm{d}t}{r}\right) - b\,(t)\,, \tag{42}$$

where

$$b\,(t) = \int\left\{\frac{s}{r^{12/11}}\exp\left(-\frac{7c_0}{11}\int\frac{\mathrm{d}t}{r}\right)\right\}\mathrm{d}t \tag{43}$$

and the solutions (370 and (38) for $u\,(x,t)$ and $a\,(x,t)$ still apply. However the solution for $f\,(x,t)$ now becomes:

$$f\,(x,t) = \left[\frac{\omega r_t}{11} + F\,(\omega) + h\,(t)\right]r^{-10/11}\exp\left(\frac{7c_0}{11}\int\frac{\mathrm{d}t}{r}\right), \tag{44}$$

where

$$h\,(t) = \frac{r_t + 7c_0}{11}b + rb_t\,. \tag{45}$$

The resulting ordinary differential equation is once again (41).

3.4 The case $r = 0$, $f \neq 0$

In the following only the non-trivial case $c_0 \neq 0$ is considered. Equations (21) to (23) may be integrated immediately to give the following solutions:

$$u\,(x,t) = m\,(11s + 7c_0x)^{4/7}\;, \tag{46}$$

$$a\,(x,t) = n\,(11s + 7c_0x)^{4/7}\;, \tag{47}$$

$$f\,(x,t) = p\,(11s + 7c_0x) - \frac{xs_t}{7c_0}\,, \tag{48}$$

where the relationship between the functions $m = m\,(t)$, $n = n\,(t)$ and $p = p\,(t)$ may be found upon substitution of equations (46) to (48) into (1). The following equation holds:

$$m_t = -11c_0mp - n + \frac{704c_0^4m^8}{5}\,. \tag{49}$$

4 Results of the non-classical analysis

In this section consideration is given the non-classical approach which is a generalization of the classical Lie method due to Bluman & Cole [6] that in corporates the surface invariant condition.

In the following the ice–sheet equation will be considered in the form (1) and the symmetry generator will now have the form:

$$\mathcal{X} = \eta_1\,(x,t,u)\,\frac{\partial}{\partial x} + \eta_2\,(x,t,u)\,\frac{\partial}{\partial t} + \phi\,(x,t,u)\,\frac{\partial}{\partial u} \tag{50}$$

and the condition for invariance of (1) is the equation

$$\mathcal{X}_E^{(2)}\,(\Psi)\,|_{\Psi=0,\Omega=0} = 0\,, \tag{51}$$

where the second prolongation operator $\mathcal{X}_E^{(2)}$ is written in the form

$$\mathcal{X}_E^{(2)} = \mathcal{X} + \phi^{[t]}\frac{\partial}{\partial u_t} + \phi^{[x]}\frac{\partial}{\partial u_x} + \phi^{[xx]}\frac{\partial}{\partial u_{xx}}\,, \tag{52}$$

where $\phi^{[t]}$, $\phi^{[x]}$ and $\phi^{[xx]}$, are defined through the transformations of the partial derivatives of u. In particular:

$$\begin{aligned}\bar{u}_{\bar{x}} &= u_x + \varepsilon\phi^{[x]}\,(x,t,u) + O\left(\varepsilon^2\right)\,,\\ \bar{u}_{\bar{t}} &= u_t + \varepsilon\phi^{[t]}\,(x,t,u) + O\left(\varepsilon^2\right)\,,\\ \bar{u}_{\bar{x}\bar{x}} &= u_x + \varepsilon\phi^{[xx]}\,(x,t,u) + O\left(\varepsilon^2\right)\end{aligned} \tag{53}$$

and the condition for invariance of surface $u{=}u\,(x,t)$ is:

$$\Omega{=}\phi{-}\eta_1u_x - \eta_2u_t = 0\,. \tag{54}$$

With the aid of Macsyma program symmgrp.max adapted for non-classical analysis it may be shown that equation (1) has the following infinitesimals:

$$\eta_2\,(x,t,u) = 1\,, \tag{55}$$

$$\eta_1\,(x,t,u) = h + x\frac{(2n+1)g^2 - g_t}{(3n+2)g}\,, \tag{56}$$

$$\phi\,(x,t,u) = u\frac{(n+1)g^2 + g_t}{(3n+2)g}\,, \tag{57}$$

where $g = g(t)$, $h = h(t)$. For this case the functions $u(x,t)$, $a(x,t)$ and $f(x,t)$ satisfy:

$$\left[(3n+2)gh + x\big((2n+1)g^2 - g_t\big)\right] u_x + (3n+2)gu_t = u\left((n+1)g^2 + g_t\right), \qquad (58)$$
$$\left[(3n+2)gh + x\big((2n+1)g^2 - g_t\big)\right] a_x + (3n+2)ga_t = (n+1)\left(g^2 + g_t\right), \qquad (59)$$

$$\begin{aligned}&\left[(3n+2)gh + x\big((2n+1)g^2 - g_t\big)\right] f_x + (3n+2)gf_t = \\ &\quad = f\left(g^2(1+2n) + g_t(1+3n)\right) + (3n+2)\left(gh_t - hg_t\right) - xg\left(\frac{g_t}{g}\right)_t. \end{aligned} \qquad (60)$$

We observe that equations (58) to (60) are essentially the same as (21) to (23) and so conclude that the non-classical symmetries are equivalent to the potential cases.

5 Particular example

Consider the case of a non-sliding ice sheet at the base so that $f = 0$ and consider the particular values, $c_0 = -0.1$, $c_1 = 1$, $c_2 = 1$ and $c_3 = 0$ with the initial condition for the ice sheet profile:

$$u(x,0) = \psi(\omega) = \tfrac{1}{2}\cos\left(\tfrac{\omega}{4}\right). \qquad (61)$$

Then according to he subcase $7c_0 + c_1 \neq 0$, $c_1 \neq 0$ and equations (24), (25) the similarity solution is

$$\omega(x,t) = \frac{x}{(1+t)^{0.0273}}, \qquad (62)$$
$$u(x,t) = \frac{\psi(\omega)}{(1+t)^{0.1272}} \qquad (63)$$

with accumulation–ablation function given by (26) and (27) so that:

$$a(x,t) = A(\omega)(1+t)^{0.0727} \qquad (64)$$

with

$$\begin{aligned}A(\omega) = {}& 0.153\times10^{-4}\cos^4\left(\tfrac{\omega}{4}\right)\sin^4\left(\tfrac{\omega}{4}\right) - 0.916\times10^{-5}\cos^6\left(\tfrac{\omega}{4}\right)\sin^2\left(\tfrac{\omega}{4}\right) + \\ &+ 0.636\times10^{-1}\cos\left(\tfrac{\omega}{4}\right) - 0.341\times10^{-1}\omega\sin\left(\tfrac{\omega}{4}\right). \end{aligned} \qquad (65)$$

In this case the propagation fronts of the ice sheet region are are found from:

$$\psi = 0, \qquad (66)$$

so

$$x_\pm(t) = \pm 2\pi(1+t)^{0.0273} \qquad (67)$$

and the finite velocity is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, x_{\pm}(t) = \pm 0.0546\pi \left(1+t\right)^{-0.973} . \tag{68}$$

Figures $1-3$ illustrate the time evolution of the ice sheet $u\,(x,t)$ and also the accumulation–oblation function $a\,(x,t)$. This example clearly demonstrates the changes in the propagation fronts $x_{\pm}(t)$ of the ice sheet. In addition it demonstrates the critical property of the existence of a closed form solution for the accumulation–ablation function which changes sign, and is negative near the propagation fronts.

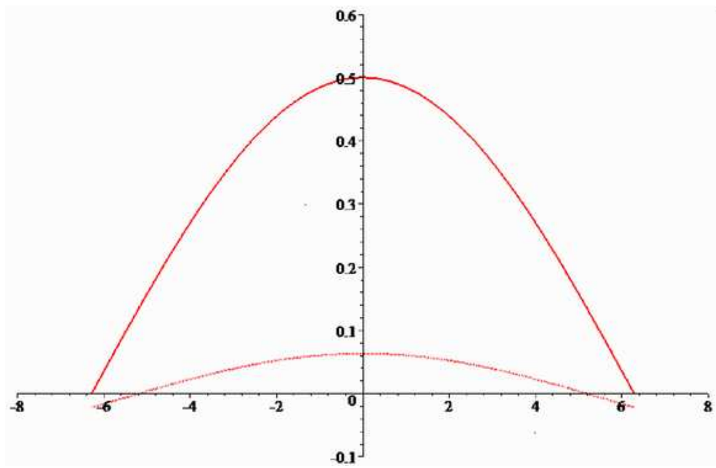

Fig. 1. Plots of the initial ice sheet profile, $u\,(x,0)$ [upper curve] and also the initial accumulation–ablation function, $a\,(x,0)$ [lower curve] versus x.

6 Comments and future work

In this paper we have concentrated on the problem of determining closed form similarity solutions using potential symmetries for the ice sheet dynamics model in the forms (1) and (2). These are the so called strong forms [2]. The main aim has been to demonstrate that classes of such solutions exist and that they contain physically realistic properties. However the strong formulation of the ice sheets dynamics problems contains certain modelling deficiencies because inadmissible solutions for which $u\,(x,t) < 0$ are possible. Certainly the solution approach presented here demonstrates the possibility of such unrealistic solutions. It is for this reason that this research is continuing and in the next phase we are seeking solutions corresponding to the weak formulation of the problem by Díaz and Schiavi [7], [8] containing a function $\beta\,(u) < 0$ and a modified model:

$$\Psi\,(x,t,u,u_t,u_x,u_{xx}) \equiv u_t - a + \beta - \left(\frac{u^{n+2}\,|u_x|^{n-1}\,u_x}{n+2} - fu\right)_x = 0\,. \tag{69}$$

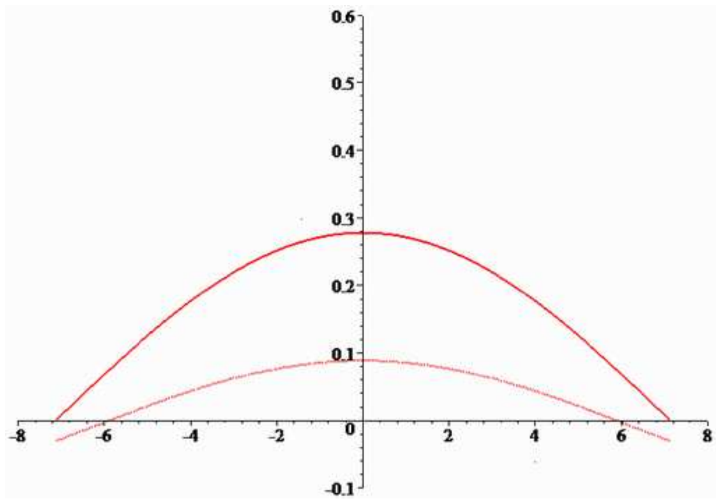

Fig. 2. Plots of the ice sheet profile, $u\,(x, 100)$ [upper curve] and also the accumulation–ablation function, $a\,(x, 100)$ [lower curve] at time $t = 100$ versus x.

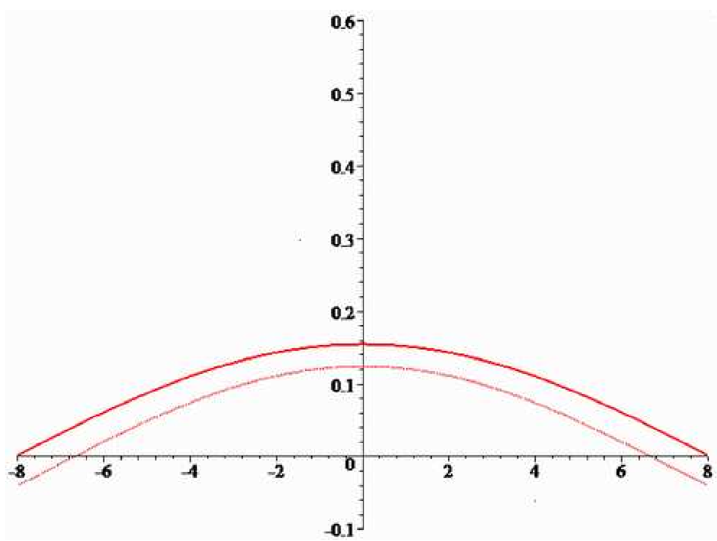

Fig. 3. Plots of the ice sheet profile, $u\,(x, 10000)$ [upper curve] and also the accumulation–ablation function, $a\,(x, 10000)$ [lower curve] at time $t = 10000$ versus x.

In this case the focus is on both a classical and a non-classical symmetry reduction of the equation. It is expected that use of the non-classical method on this occasion will extend the range of possible solutions.

In a further development we consider a more general framework to encompass a wider range of physical applicability for the mathematical analysis. This is achieved by defining two functions:

$$\phi\left(r\right)=\left|r\right|^{n-1}r\,,\quad \psi\left(s\right)=s^{n} \tag{70}$$

and defining the new functions $U(x,t)$ and $b\left(s\right)$ so that:

$$U=u^{m}=\psi\left(u\right),\ \Rightarrow\ U^{1/m}=u=\psi^{-1}\left(U\right)=b\left(u\right) \tag{71}$$

so that:

$$\phi\left(\psi\left(u\right)_{x}\right)=\phi\left(U_{x}\right)=\left|U_{x}\right|^{p-2}U_{x}\,, \tag{72}$$

where $p=n+1$. In this way the mathematical framework may be taken to be:

$$b\left(U\right)_{t}-\left[k\phi\left(W\right)-fb\left(U\right)\right]_{x}+\beta\left(U\right)-a\left(x,t\right)=0\,, \tag{73}$$

where k is a constant and:

$$W=U_{x}\,. \tag{74}$$

This may also be written in a conserved or potential form by writing

$$V_{x}+\lambda-b=0\,, \tag{75}$$

$$V_{t}-k\psi+bf=0\,, \tag{76}$$

$$W=U_{x}\,, \tag{77}$$

where $b=b\left(U\right)$, $\psi=\psi\left(U\right)$, $\lambda=\lambda\left(x,t,U\right)$, $f=f\left(x,t\right)$ provided that:

$$a-\beta\equiv\lambda_{t}\,. \tag{78}$$

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