The evolution of solutions of plane ideal plasticity

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It's well known, that the symmetries of a system of differential equations allow transforming its solutions to solutions of this system. Using this property, from two known solutions of the theory of plasticity: the solution of Nadai for circular cavity stressed by normal and shear pressure, and Prandtl's solution for a block compressed between perfectly rough plates, there were constructed new analytical exact solutions of the system of two–dimensional ideal plasticity equations.

PACS: 46.35+z; 02.20-a

Key words: plasticity, symmetry analysis, Lie group and algebra methods, exact solutions of differential equations

1 Introduction

The symmetry analysis of differential equations has many applications in an investigation of a systems of partial differential equations (PDEs). With a help of a group of symmetries it's possible to obtain some information about initial structure of PDEs, it allows to construct invariant and some others classes of exact solutions, to classify PDEs by admitting groups and so on ([1] − [8]). The main property of symmetry admitting by the system of PDEs, is that under its action any solution of the system is moved to the solution of this system. Using this property one can construct a new solutions without integrating of the given system, only by means of group transformations under known solutions. In such a way, for example, some interesting results were obtained in [5] for Kadomtsev–Pogutse equations. Note, that this way is effective one only if we have a sufficiently rich group of point transformations.

From exact solutions one can construct a family of so-called *S–solutions*, i.e obtained by means of symmetries. This family of S–solutions depends of the group parameter a, if $a = 0$ then we have an initial solution. This procedure is called the *"reproduction"* [6] or the *"evolution"* of solutions. Sometime it's possible to construct the general solution of equations from particular one only by means of point transformations, and new S–solutions are satisfying to a new boundary–value conditions.

The classical system of the plane ideal plasticity is investigated during a many years but there are a few of its exact solutions: the Prandtl's solution; the solution for a cavity of circular form, stressed by uniform pressure; the solution of Nadai for the stresses in the plastic region round a circular cavity loaded by a constant shear stress in addition to a uniform pressure; and the spiral–symmetrical solution [7].

In [8] there were constructed the analytical solutions for some boundary problems. All these solutions are widely used for a testing of a numerical calculations, allow to estimate an assurance factor of some constructions and so on. Therefore we can say that there is a lack of exact solutions of the system under consideration.

The plan of the paper is as follows. In the section 2 we give briefly some basic definitions and statements of the theory of point transformations and some known results for the system of plane ideal plasticity, which are necessary for the construction of a new exact solutions.

In the section 3 we deduce S–solution from the solution of Nadai for plastic state round a circular cavity. As the result, the obtained family of new solutions allows describing the extension of a cavities of different forms, in particular, a cavity of the form of limacon of Pascal.

In the section 4 Prandtl's solution was transformed. As result we obtained a lot of exact solutions. We selected only restricted ones along Oy–axis and these solutions can be used for the solving of boundary–value problems for compression of a blocks by rough parallel plates. We can hope, that with the help of these S–solutions it will be possible to describe not only compression of the thin blocks, but the thick ones too.

2 Point transformations

Let give some informal concepts, for more precise definitions see for example [1]. Let's consider a system of PDEs with two unknown functions $u = u(x,y)$, $v = v(x,y)$:

$$F_1(x,y,u,v) = 0\,, \quad F_2(x,y,u,v) = 0\,.$$

We can say, that this system admits a *symmetry* or an one–parameter group G of point transformations:

$$\begin{aligned} x' &= f^1(x,y,u,v,a)\,, \quad f^1|_{a=0} = x\,, \qquad y' = f^2(x,y,u,v,a)\,, \quad f^2|_{a=0} = y\,,\\ u' &= g^1(x,y,u,v,a)\,, \quad g^1|_{a=0} = u\,, \qquad v' = g^2(x,y,u,v,a)\,, \quad g^2|_{a=0} = v\,, \end{aligned}$$

if the given system of PDEs has the same form for the new variables x', y', u', v'. An infinitesimal operator

$$X = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v}$$

corresponds to the point transformations if its coordinates are satisfying to the Lie equations

$$\begin{aligned} \frac{\mathrm{d}x'}{\mathrm{d}a} &= \xi^1\,, \quad x'|_{a=0} = x\,, \qquad \frac{\mathrm{d}y'}{\mathrm{d}a} = \xi^2\,, \quad y'|_{a=0} = y\,,\\ \frac{\mathrm{d}u'}{\mathrm{d}a} &= \eta^1\,, \quad u'|_{a=0} = u\,, \qquad \frac{\mathrm{d}v'}{\mathrm{d}a} = \eta^2\,, \quad v'|_{a=0} = v\,. \end{aligned}$$

So, under admitting point transformations any solution of the initial system of PDEs $u = u(x, y)$, $v = v(x, y)$ is moved to the solution of the form $u' = u'(x', y')$, $v' = v'(x', y')$ which we can name as S*–solution*.

Let's consider the classic system of plane ideal plasticity with von Mises' criterion [9]:

$$\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0\,, \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0\,, \\ &(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2\,, \end{aligned} \tag{1}$$

where σ_x, σ_y, τ_{xy} are components of a stress tensor, k is a constant of plasticity.

By means of changing of variables

$$\begin{aligned} \sigma_x &= \sigma - k \sin 2\theta\,, \\ \sigma_y &= \sigma + k \sin 2\theta\,, \\ \tau_{xy} &= k \cos 2\theta \end{aligned}$$

the system (1) is reduced to

$$\begin{aligned} &\frac{\partial \sigma}{\partial x} - 2k \left(\frac{\partial \theta}{\partial x} \cos 2\theta + \frac{\partial \theta}{\partial y} \sin 2\theta \right) = 0\,, \\ &\frac{\partial \sigma}{\partial y} - 2k \left(\frac{\partial \theta}{\partial x} \sin 2\theta - \frac{\partial \theta}{\partial y} \cos 2\theta \right) = 0\,, \end{aligned} \tag{2}$$

where σ is a hydrostatic pressure, $\theta + \pi/4$ is the angle between the first principal direction of a stress tensor and the Ox–axis.

It's known [3], that the system (2) admits the infinite group of symmetries. Its subgroup of point transformations is formed by following generators:

$$\begin{aligned} X_1 &= \partial_x\,, \quad X_2 = \partial_y\,, \quad X_3 = x\partial_x + y\partial_y\,, \\ X_4 &= -x\partial_y + y\partial_x + \partial_\theta\,, \quad X_5 = \partial_\sigma\,, \\ X_6 &= \xi_1 \partial_x + \xi_2 \partial_y + 4k\theta \partial_\sigma - \frac{\sigma}{k}\,\partial_\theta\,, \\ X &= \xi \partial_x + \eta \partial_y\,, \end{aligned} \tag{3}$$

where

$$\xi_1 = -x \cos 2\theta - y \sin 2\theta - y\frac{\sigma}{k}\,, \quad \xi_2 = y \cos 2\theta - x \sin 2\theta + x\frac{\sigma}{k}\,,$$

and (ξ, η) is an arbitrary solution of the following lineal system of equations:

$$\begin{aligned} &\frac{\partial \xi}{\partial \theta} - 2k \left(\frac{\partial \xi}{\partial \sigma} \cos 2\theta + \frac{\partial \eta}{\partial \sigma} \sin 2\theta \right) = 0\,, \\ &\frac{\partial \eta}{\partial \theta} - 2k \left(\frac{\partial \xi}{\partial \sigma} \sin 2\theta - \frac{\partial \eta}{\partial \sigma} \cos 2\theta \right) = 0\,. \end{aligned} \tag{4}$$

Transformations, which are corresponded for any generator of (3), convert the system (2) to itself. Thus, for the generator X we have transformations of independent variables:

$$x' = x + a\xi\,, \quad y' = y + a\eta\,, \tag{5}$$

where a is an arbitrary sufficiently small parameter, (ξ, η) is an arbitrary solution of (4). Because of X is admitting generator, hence any solution of the system (2) is transformed to the solution of this system under the action of transformations (5). Note, that we've shown the variables that are changing under this transformation, other variables are omitted.

3 Reproduction of solution for a circular cavity

The system (1) in polar co-ordinates (r, ϕ) has the form

$$\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{\sigma_r - \sigma_\phi}{r} = 0\,,\\
&\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\phi}{\partial \phi} + \frac{2\tau_{r\phi}}{r} = 0\,,\\
&(\sigma_r - \sigma_\phi)^2 + 4\tau_{r\phi}^2 = 4k^2\,.
\end{aligned} \tag{6}$$

Let's consider the solution of Nadai [9]

$$\begin{aligned}
&\left.\begin{matrix}\sigma_r\\ \sigma_\phi\end{matrix}\right\} = -p + k\ln\frac{r^2 + \sqrt{r^4 - m^2R^4}}{R^2(1+\sqrt{1-m^2})} + k\left(\sqrt{1-m^2} \mp \frac{1}{r^2}\sqrt{r^4 - m^2R^4}\right),\\
&\tau_{r\phi} = km\frac{R^2}{r^2}\,, \quad m = \frac{t}{k}\,.
\end{aligned} \tag{7}$$

This solution describes a plastic state round a circular cavity of radius R, situated in an infinite medium loaded by uniformly distributed pressure p, and tangent stress $\tau_{r\phi}$ is equal to t:

$$\sigma_r\big|_{r=R} = -p\,, \quad \tau_{r\phi}\big|_{r=R} = t\,.$$

In cartesian system of co-ordinates this solution has the form:

$$\begin{aligned}
\theta(x,y) &= \arctan\frac{y}{x} + \frac{1}{2}\arccos\frac{mR^2}{x^2+y^2}\,,\\
\sigma(x,y) &= -p + k\ln\frac{x^2+y^2+\sqrt{(x^2+y^2)^2 - m^2R^4}}{R^2(1+\sqrt{1-m^2})} + k\sqrt{1-m^2}\,.
\end{aligned} \tag{8}$$

Let's take the solution of the system (4) in the form

$$\begin{aligned}
\xi &= \alpha\sigma + k(\alpha\sin 2\theta - \beta\cos 2\theta) + C_1\,,\\
\eta &= \beta\sigma - k(\alpha\cos 2\theta + \beta\sin 2\theta) + C_2\,.
\end{aligned}$$

For a simplicity, let $\alpha = 1$, $\beta = 0$, $C_1 = p$, $C_2 = 0$:

$$\begin{aligned} \xi &= \sigma + k\sin 2\theta + p\,, \\ \eta &= -k\cos 2\theta\,, \end{aligned} \tag{9}$$

then transformations (5) will take the form:

$$\begin{aligned} x' &= x + a(\sigma + k\sin 2\theta + p)\,, \\ y' &= y - ak\cos 2\theta\,. \end{aligned} \tag{10}$$

So, in this case, S–solution obtained from (8) will have the form $\theta = \theta(x', y')$, $\sigma = \sigma(x', y')$.

In important practical problems, when the tangent stress $\tau_{r\phi}$ is equal to zero:

$$\sigma_r\big|_{r=R} = -p\,, \quad \tau_{r\phi}\big|_{r=R} = t = 0\,.$$

the solution of Nadai (7) has the form

$$\begin{aligned} \sigma_r &= -p + 2k\ln\frac{r}{R}\,, \\ \sigma_\phi &= -p + 2k\left(1 + \ln\frac{r}{R}\right)\,, \\ \tau_{r\phi} &= 0\,. \end{aligned} \tag{11}$$

In cartesian system of co-ordinates solution (11) is the following:

$$\begin{aligned} \theta &= \arctan\frac{y}{x} + \frac{\pi}{4}\,, \\ \sigma &= -p + k + k\ln\frac{x^2 + y^2}{R^2}\,. \end{aligned} \tag{12}$$

Using (10) and the first equation of (12) we can write

$$\begin{aligned} \cos 2\theta &= -\frac{2x'y'}{x'^2 + y'^2} = -2\frac{(x + a(\sigma + k\sin 2\theta + p))(y - ak\cos 2\theta)}{(x + a(\sigma + k\sin 2\theta + p))^2 + (y - ak\cos 2\theta)^2}\,, \\ \sin 2\theta &= \frac{x'^2 - y'^2}{x'^2 + y'^2} = \frac{(x + a(\sigma + k\sin 2\theta + p))^2 - (y - ak\cos 2\theta)^2}{(x + a(\sigma + k\sin 2\theta + p))^2 + (y - ak\cos 2\theta)^2}\,. \end{aligned}$$

Resolving the ultimate system with respect to $\cos 2\theta$, $\sin 2\theta$ we obtain:

$$\begin{aligned} \cos 2\theta &= -2\frac{y(x + a(\sigma + p - k))}{(x + a(\sigma + p - k))^2 + y^2}\,, \\ \sin 2\theta &= \frac{(x + a(\sigma + p - k))^2 - y^2}{(x + a(\sigma + p - k))^2 + y^2}\,. \end{aligned}$$

In order to obtain S–solution, we have to substitute transformed variables x', y' to the solution (12). Therefore, taking into account (10), the function θ will take the form:

$$\theta = \arctan\frac{y'}{x'} + \frac{\pi}{4} = \arctan\frac{y}{x + a(\sigma + p - k)} + \frac{\pi}{4}\,. \tag{13}$$

For the function σ from the second expression of (12) we obtain:

$$\sigma = -p + k - k \ln R^2 + k \ln \frac{\left[y^2 + (x + a(\sigma + p))^2 - a^2 k^2\right]^2}{y^2 + (x + a(\sigma + p - k))^2}\,. \tag{14}$$

The S–solution (13), (14) in the polar system of co-ordinates has the form

$$\begin{aligned} \theta &= \arctan \frac{r \sin \phi}{r \cos \phi + a(\sigma + p - k)} + \frac{\pi}{4}\,, \\ \sigma &= -p + k - k \ln R^2 + \\ &+ k \ln \frac{\left[r^2 + 2ar(\sigma + p) \cos \phi + a^2((\sigma + p)^2 - k^2)\right]^2}{r^2 + 2ar(\sigma + p - k) \cos \phi + a^2(\sigma + p - k)^2}\,. \end{aligned} \tag{15}$$

The S–solution (15) describes plastic state round a cavity of the form $r = R - 2ak \cos \phi$ (i.e. a form of limacon of Pascal), with boundary conditions:

$$\sigma\big|_{r = R - 2ak \cos \phi} = -p + k\,, \quad \theta\big|_{r = R - 2ak \cos \phi} = \phi + \pi/4\,,$$

where a is an arbitrary parameter. From a mechanical sense the value of parameter R should be more or equal to $2ak$. If $a = 0$, then we have the initial solution (12). If instead of (9) we'll take other solution of the system (4) with other values of α, β, C_1, C_2, then we'll obtain different forms of limacon of Pascal for cavities.

4 Reproduction of Prandtl's solution

Let's write the Prandtl's solution [9] of the system (2) in the form

$$\sigma = -kx + k\sqrt{1 - y^2}\,, \quad 2\theta = \arccos y\,. \tag{16}$$

The equations of the corresponding families of slip lines are the following:

$$\begin{aligned} x &= -2\theta + \sin 2\theta + K_1\,, \quad & y &= \cos 2\theta\,, \\ x &= 2\theta + \sin 2\theta + K_2\,, \quad & y &= \cos 2\theta\,. \end{aligned}$$

Under the transformations (5), the Prandtl's solution will transform to the following one:

$$\sigma = -kx + k \sin 2\theta + ak\xi\,, \quad y = \cos 2\theta + a\eta\,, \tag{17}$$

and the slip lines will have the form:

$$x = \mp 2\theta + a\xi + \sin 2\theta + K_i\,, \quad y = \cos 2\theta + a\eta\,, \quad i = 1,\, 2\,.$$

Therefore, in order to obtain new solutions, we have to find out some solutions of the system (4),which we can seek in the forms:

$$\xi = \alpha\sigma + F(\theta)\,, \qquad \eta = \beta\sigma + G(\theta)\,; \tag{18}$$

$$\xi = f(\theta) \exp\left(\frac{\sigma}{2k}\right), \qquad \eta = g(\theta) \exp\left(\frac{\sigma}{2k}\right); \tag{19}$$

$$\xi = \alpha(\theta) \sin \frac{\sigma}{2k} + \beta(\theta) \cos \frac{\sigma}{2k}\,, \qquad \eta = F(\theta) \sin \frac{\sigma}{2k} + G(\theta) \cos \frac{\sigma}{2k}\,; \tag{20}$$

$$\xi = \alpha(\theta) \sinh \frac{\sigma}{2k} + \beta(\theta) \cosh \frac{\sigma}{2k}\,, \quad \eta = F(\theta) \sinh \frac{\sigma}{2k} + G(\theta) \cosh \frac{\sigma}{2k}\,, \tag{21}$$

where functions f, g, α, β, F, G are determined as a solution of corresponding systems of ODEs.

It's easy to see, that under the transformation (18) the Prandtl's solution is moved to the solution of the same form (there is no new solution). Under the transformations (19), (21) the solution will be transform to the no restricted one along the Oy–axis, hence we don't consider solutions of this form.

Let's consider the action of ultimate transformation (20) to the slip lines. Substituting (20) to the system (4) we obtain the system of equations for a determination of functions α, β, F, G

$$\begin{aligned} &\alpha' + \beta\cos 2\theta + G\sin 2\theta = 0\,, \quad &\beta' - \alpha\cos 2\theta - F\sin 2\theta = 0\,,\\ &F' + \beta\sin 2\theta - G\cos 2\theta = 0\,, \quad &G' - \alpha\sin 2\theta + F\cos 2\theta = 0\,. \end{aligned} \tag{22}$$

Let's derive each of the equations (22) with respect to θ. Then, according to (22), we obtain the equations:

$$\begin{aligned} &\alpha'' + \alpha + 2F' = 0\,, \quad &F'' + F - 2\alpha' = 0\,,\\ &\beta'' + \beta + 2G' = 0\,, \quad &G'' + G - 2\beta' = 0\,. \end{aligned} \tag{23}$$

The real solution for α, F has the form

$$\begin{aligned} \alpha &= C_1\cos(1\pm\sqrt{2})\theta - C_2\sin(1\pm\sqrt{2})\theta\,,\\ F &= C_1\sin(1\pm\sqrt{2})\theta + C_2\cos(1\pm\sqrt{2})\theta\,, \end{aligned}$$

where C_1, C_2 are arbitrary constants.

Then, solution for β, G of the second system (23), according to (22) will take the form:

$$\begin{aligned} \beta &= (1\pm\sqrt{2})\left(C_2\cos(1\mp\sqrt{2})\theta - C_1\sin(1\mp\sqrt{2})\theta\right),\\ G &= (1\pm\sqrt{2})\left(C_2\sin(1\mp\sqrt{2})\theta + C_1\cos(1\mp\sqrt{2})\theta\right). \end{aligned}$$

For a simplicity let's $C_1 = 1$, $C_2 = 0$, and let take the lower sign, then (20) we can write down in the form:

$$\begin{aligned} \xi &= \cos(1-\sqrt{2})\theta\sin\frac{\sigma}{2k} - (1-\sqrt{2})\sin(1+\sqrt{2})\theta\cos\frac{\sigma}{2k}\,,\\ \eta &= \sin(1-\sqrt{2})\theta\sin\frac{\sigma}{2k} + (1-\sqrt{2})\cos(1+\sqrt{2})\theta\cos\frac{\sigma}{2k}\,. \end{aligned}$$

The transformations of the slip lines can be written as follows:

$$\begin{aligned} x &= \mp 2\theta + a\Big[\cos(1-\sqrt{2})\theta\sin(\pm\theta + c_i) -\\ &\quad -(1-\sqrt{2})\sin(1+\sqrt{2})\theta\cos(\pm\theta + c_i)\Big] + \sin 2\theta + K_i\,,\\ y &= \cos 2\theta + a\Big[\sin(1-\sqrt{2})\theta\sin(\pm\theta + c_i) +\\ &\quad +(1-\sqrt{2})\cos(1+\sqrt{2})\theta\cos(\pm\theta + c_i)\Big], \qquad i = 1, 2\,. \end{aligned} \tag{24}$$

From relations (24) it's follow that new solution has restricted slip–lines, and we can use these solutions for description of plastic flow of block compressed by two rough plates.

With the increment of parameter a the thickness of block increases and becomes approximately equal to $2(h + a)$, where h is a thickness of the initial block. The new S–solution obtained from the Prandtl's solution (16) by means of symmetry transformations (5), has the form:

$$\sigma = -kx + a\left[\cos(1-\sqrt{2})\theta \sin\frac{\sigma}{2k} - (1-\sqrt{2})\sin(1+\sqrt{2})\theta\cos\frac{\sigma}{2k}\right] + k\sqrt{1-y^2}\,,$$
$$y = \cos 2\theta + a\left[\sin(1-\sqrt{2})\theta \sin\frac{\sigma}{2k} + (1-\sqrt{2})\cos(1+\sqrt{2})\theta\cos\frac{\sigma}{2k}\right],$$

where a is an arbitrary parameter. For large values of parameter a we have a S–solution far different from Prandtl's solution. It can be used for analysis of plastic flows for thick compressed block.

This research was partially supported by PROMEP (Mexico).

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