

\mathcal{PT} –symmetry, ghosts, supersymmetry and Klein–Gordon equation

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Parallels between the concepts of symmetry, supersymmetry and (recently introduced) \mathcal{PT} –symmetry are reviewed and discussed, with particular emphasis on the new insight in quantum mechanics which is rendered possible by their combined use.

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1 Introduction

When we return to the early stages of development of quantum mechanics we reveal that the fascination of its authors by their new discoveries must have been enormous. Suddenly, they were able to resolve many old and tough puzzles like the “incomprehensible” and mysterious stability of atoms with respect to an expected steady radiation of their moving electrons. Under the name of \mathcal{PT} –symmetry [1], perhaps, we might easily be witnessing a certain continuation of these discoveries in the nearest future.

As we all know, the “trick” of the founders of quantum mechanics was virtually elementary and consisted in a suitable replacement of any classical observable quantity by an appropriate essentially self-adjoint operator in Hilbert space \mathcal{H} (i.e., Hamiltonian $H = H^\dagger$ for the energy, etc). Unfortunately, during the subsequent applications of quantum mechanics, a few paradoxes emerged in the early forties, especially in connection with relativistic problems (cf., e.g., the “Dirac’s sea” [2]), representing an important practical limitation of the whole trick, and hinting that the “realistic” Hamiltonians might be non-Hermitian.

This discovery forced the phenomenologically oriented physics community to forget about the optimism and ambitions of the early thirties. Theoreticians stampeded towards “certainties” offered by the half-explored territory of relativistic quantum field theory. Such a controversial situation seems to have survived more than half a century and, admittedly, it forms also an important part of what we are going to discuss.

In a way inspired by the pioneering letter [1] we believe that the generic quantization recipe may be perceived as only too much innovation–resistant. Thus, we are going to review here a few aspects of the Bender’s and Boettcher’s generalization of the quantization recipe and of some of its particular applications in the context of (super)symmetries.

Perhaps, our supplementary motivation lies in the observation that in its time, the very idea of the quantization was truly revolutionary. Still, it arose a wave

of protests among conservative physicists of all professional qualities (let us just mention their most famous EPR branch [3]). Hopefully, the similar protests against \mathcal{PT} -symmetry, however existing [4], will prove much less resistant, the more so after the emergence of its defenses and reviews like the one which follows.

2 Why \mathcal{PT} -symmetry?

Only towards the end of the second millennium, Carl Bender dared to return from field theory to quantum mechanics and formulated his (nowadays, almost famous) “but wait a minute!” project [5] where, basically, he advocated the necessity of a tentative weakening of the (mathematically “unnecessarily strong”) Hermiticity requirement for the observables. In the influential letter [1], this type of “heresy” was formulated as inspired by discussions with their predecessors and colleagues. In a broader historical perspective, its implicit origins may be traced back to several independent formal studies and/or isolated comments reflecting needs of several branches of phenomenological physics and occurring (or rather “lost”, here and there) in the more mathematically oriented literature concerning, mostly, certain peculiar non-Hermitian anharmonic oscillators with real spectra [6].

It is worth noting at this point that the same idea appeared, independently, a couple of years sooner, in the fully separate contexts of nuclear physics (under the name of quasi-Hermiticity [7]) and in a few other contexts mentioned in [1]. Concerning the present state of art, interested reader may generate easily a comparatively complete set of relevant citations when looking in the proceedings of the (up to now, two) dedicated international conferences [8].

There is probably not yet time for an adequate and critical evaluation of the resulting new (though not yet ripe enough) formulation of the innovated, so called \mathcal{PT} -symmetric [1, 9] (or, if you wish, quasi-Hermitian [7]) or \mathcal{QPT} -symmetric [10] or pseudo-Hermitian [11] or \mathcal{CPT} -symmetric [12, 13] quantum mechanics. I may only note that the use of all these nicknames for the same theory is, by my own opinion, redundant. That’s why I am sticking to the Bender’s terminology (\mathcal{PT} -symmetric quantum mechanics, PTQM), understanding that \mathcal{P} and \mathcal{T} need not necessarily mean just the parity and time reversal, respectively.

In particular, \mathcal{T} is to be read as an abbreviation for the more rigorous mathematical requirement of “being essentially self-adjoint in \mathcal{H} ” [14], so that the time-reversal symmetry (i.e., the commutativity of the Hamiltonian with the antilinear operator \mathcal{T}) might find one of its most frequent applications in the very formal definition of the standard Hermitian-conjugation mapping, $\mathcal{T}HT \equiv H^\dagger$.

2.1 A brief recollection of a few much older relevant works

In the previous paragraph, we may pre-multiply operator \mathcal{T} by an indefinite involution \mathcal{P} and arrive at the explicit definition of the \mathcal{PT} -symmetry. All the generalized non-involutory and non-diagonal though, necessarily, non-singular and essentially self-adjoint [7] forms of our \mathcal{P} are also admissible and one thus returns just to the old Dirac’s works [15] with his pseudo-metric η simply replaced by \mathcal{P} .

In this spirit, the above-mentioned retreat to field theory was a pure misunderstanding. Everybody knows that this theory (with all its infinitely many degrees of freedom and infinitely large renormalizations etc) represents a not too user-friendly key to our understanding of microscopic systems. Curiously enough, even the Dirac, see problem itself, expelled carefully from before the door of quantum mechanics (and, formally, reducible to an indefiniteness of the “physical” Dirac’s pseudo-metric η in the full Hilbert space) returned through the window of field theory. It keeps living there, e.g., as a lasting difficulty with ghosts (cf. the Pauli’s and Gupta’s and Bleuler’s studies [16] and the Lie–Wick’s and Nakanishi’s discussions [17] or their recent revitalization [18]).

3 How can we survive with indefinite metric?

Once the rule $H^\dagger\eta = \eta H$ of the pseudo-Hermiticity with a Hermitian “Hilbert-space-metric-operator” $\eta = \eta^\dagger \neq \eta_{\text{traditional}} = I$ becomes needed in field theory, it is possible to say that “the appearance of negative probability is the greatest problem in the indefinite metric theory” while “the great physicists proposed wrong resolution of it” [19]. A key to the solution of this problem has been offered, very recently, by Ali Mostafazadeh [11] who paid very detailed attention to one of the most common and physical \mathcal{PT} -symmetric models, see to Klein–Gordon equation in its Feshbach–Villars form [20]. He imagined that *every* given and fixed \mathcal{PT} -symmetric Hamiltonian operator H may, in general, satisfy the necessary pseudo-Hermiticity intertwining rule with *many different* pseudo-metrics η_m . This simple idea (revealed, independently and practically in parallel, by several groups of other authors [7, 12, 21]) initiated a new wave of development of the theory because some of the new metrics may be positive (i.e., $\eta = \eta_+ > 0$ in the notation of ref. [11]).

The first step of all the applications of the new theory should lie, therefore, in a carefully explained transition from the “simple” or “initial” indefinite metric (let us call it \mathcal{P}) to the “correct” or rather “physical” alternative metric $\eta_+ > 0$.

A compact denotation $\eta_+ \equiv \mathcal{CP}$ used in ref. [12] looks to the present author as one of the best conventions on the market, with one of his reasons being that the “charge” operator C coincides with his own (and still older) quasi-parity Q (cf. [22] dating back to 1999). An even older paper by Scholtz, Geyer and Hahne [7] should be still more decisively recalled as another recommended reading. In this reference, the positive $\eta_+ \neq I$ were also already known and studied and the related H (be it Hermitian or not in the usual sense) has been called quasi-Hermitian (this means Hermitian in the nontrivial, non-isotropic metric \mathcal{CP}) there.

In such a context, the ghosts of the field theories [23] emerge as defined as states with the vanishing pseudo-norm defined with respect to the indefinite metric (i.e., pseudo-metric) \mathcal{P} . One may then understand the S -matrix as an operator which is unitary in the sense of the indefinite metric [24]. It is not genuinely unitary because the norm positivity is not guaranteed with respect to \mathcal{P} [19]. Still, after the change of the metric (i.e., after the re-construction of the Hamiltonian-dependent quasi-Hermiticity by a re-definition of the inner product with respect to the positively

definite metric \mathcal{CP}) the physical appearance of the negative-norm states becomes forbidden.

Now, our key message may be formulated as a statement that the physical interpretation of the ghosts need not necessarily proceed solely in the traditional Gupta–Bleuler–like ghost–elimination spirit, i.e., in a way based on an appropriate specification of the “physical subspace”. The original Hilbert space \mathcal{H} need not be necessarily declared overcomplete on the physical grounds, and some subsidiary conditions need not necessarily be imposed.

Indeed, in the new metric \mathcal{CP} (which is to be declared “correct and physical” and which is, in principle, ambiguous and dynamically dependent on H), one can alternatively get rid of the interpretation difficulties via the use of the new norm.

4 Why \mathcal{PT} –symmetric supersymmetry?

4.1 Why supersymmetry at all?

One of the main motivations of the present text originates from the well known key to the exactly solvable 1D models found in their supersymmetric (SUSY) re-interpretation [25]. It is amusing to recollect that such an application of SUSY emerged, historically speaking, as a quite unexpected byproduct of the originally more ambitious supersymmetric quantum mechanics (SUSYQM) of Ed Witten.

We consider the latter formalism worth extending to the \mathcal{PT} –symmetric world. As long as the \mathcal{PT} –symmetry property itself is not too dissimilar from its “Hermiticity” predecessor $H = H^\dagger$, one may expect that an active use of \mathcal{PT} –symmetric models could be capable of altering the present status and the role of all the standard symmetries in general and of the supersymmetry in particular.

To set the scene, let us return to the simple linear parity operator \mathcal{P} and to an eigenvalue problem $H|\Psi\rangle = E|\Psi\rangle$ with symmetry $H\mathcal{P} = \mathcal{P}H$. In the light of Schur’s lemma this implies that every linear combination of $|\Psi\rangle$ and $\mathcal{P}|\Psi\rangle$ will also satisfy the same eigenvalue problem so that in the most common one–dimensional and non-degenerate (= Sturm–Liouville) setting we may immediately classify all the solutions $|\Psi\rangle$ according to their parity.

Once we move to the more sophisticated symmetries, the same procedure makes our spectra multiply indexed. One of the most successful applications of such a strategy may be undoubtedly found in the physics of elementary particles where the symmetries of the interactions proved to be a powerful source of the classification of the possible solutions (= particle multiplets).

Paradoxically enough, SUSY as a mathematically most natural transition to the symmetries between the bosons and fermions (called, in mathematics, graded algebras) failed in practice. No SUSY–partner element of any supermultiplet has been found up to now. At the same time, the use of the first nontrivial graded Lie algebra $sl(1|1)$ proved extremely fruitful within SUSYQM. In Hermitian case, its three ‘graded’ generators (see Hamiltonian H and the two “supercharges Q and P satisfying there the ‘fermionic’ nilpotence rule $PP = QQ = 0$ plus a compatibility commutation relations $HP - PH = HQ - QH = 0$) are described in detail, say, in

review paper [25].

4.2 Why supersymmetry without Hermiticity?

An explicit sample of the modified SUSYQM in its various \mathcal{PT} -symmetric versions may be found among papers [26]. Probably the first application of such a non-Hermitian quantum-mechanical SUSY formalism to the fully and exactly solvable (spiked) harmonic oscillator may be found in ref. [27]. At this point it would be useful to carbon-copy some formulae for illustration but we must skip such a plan due to the absolute shortage of space for such a purpose.

In the middle of our very concise discussion of the subject, surviving without the formulae, it is still necessary to emphasize that \mathcal{PT} -symmetry is anti-linear (i.e., non-linear) so that all the apparent Schur-type parallels with the ordinary linear symmetries (and SUSY) will immediately break down [5]. Similar formal challenges accelerated, after all, the very recent development of the field. I.a., it has been found that a much easier access to many relevant structures immanent in the majority of the “weakly Hermitian” \mathcal{PT} -symmetric models (and, in particular, to their specific spectral representations [21, 28] and/or to the existence of the very specific, so called exceptional points in their spectra [29]) may be mediated by many much more elementary models [30].

This further supported our interest in SUSY constructions in a way which applies, in particular, to the really exceptional exactly solvable harmonic oscillator limit of virtually all of the above-mentioned peculiar anharmonic oscillators [22]. In the same direction, the first successful steps have been made also towards the more-particle \mathcal{PT} -symmetric exactly-solvable models of the Calogero [31] or Winternitz [32] types. In this context, many questions still remain open [33].

5 \mathcal{PT} form of SUSY in application to Klein–Gordon equation

In an overall non-Hermitian setting, the SUSY Hamiltonian H becomes a direct sum of its diagonal ‘left’ and ‘right’ Hamiltonian-type sub-operators $H_{(L,R)}$. In the same two-by-two partitioned notation (cf. [25] for all details) both Q and P are, respectively, lower and upper triangular two-by-two matrices. They contain just one off-diagonal element (say, operators a and c , respectively). Of course, the standard Hilbert-space representations of the latter a (annihilation operator) and c (creation operator) are usually non-diagonal and may be often written in the one-diagonal upper- and lower-triangular infinite-dimensional matrix form, respectively.

In our recent paper [34] we asked what happens if one relaxes the standard Hermiticity requirements. What we did in ref. [34] was, in essence, just a transfer of the underlying SUSY-type factorization of the Hamiltonian to the domain of the Klein–Gordon-type equations. Interested reader is recommended to search for the explicit formulae in *loc. cit.*

Basically, we proceeded in full analogy with the non-relativistic case. Keeping \mathcal{P} equal to the (most elementary operator of) parity we recollected the standard

procedure (seen as a source of interest in the imaginary cubic anharmonicities $-ix^3$ in the late seventies) and assumed that any suitable preselected spatially symmetric and real (read: \mathcal{T} -symmetric) potential is made non-Hermitian and \mathcal{PT} -symmetric by adding another, purely imaginary and spatially antisymmetric component to it. A deeper understanding of the similar models can already rely on the intensive technical developments in the field. Thus, explicit constructions may be facilitated by a recourse to the delta-expansion techniques of field theory [35] or to the WKB [36] and strong-coupling [37] perturbation expansions or to the quasi-exact solution techniques [38]. Last but not least, the use of the language of Bethe ansatz might prove also an enormously efficient tool [39].

6 Conclusions

6.1 Practical use of \mathcal{PT} -symmetry

We emphasized that once we accept the necessity of a Lorentz covariance of some realistic dynamical equations in quantum setting, we arrive at one of the oldest, most physical and most popular PTQM example representing the Klein-Gordon version of the relativistic quantum mechanics in the form authored by Feshbach and Villars in the middle of fifties [20], clarified to be consistent with the standard postulates of quantum mechanics by Ali Mostafazadeh [40] and encountering an unexpected resurrection in cosmology at present [41].

In this model, a successful combination of the Lie (= linear) symmetry with the non-Hermiticity of the generator H_{FV} of the time evolution exemplifies a specific non-diagonal and two-by-two partitioned form of the indefinite metric \mathcal{P} or η .

Classical representation theory did find a place for both the linear and antilinear operators (cf. reviews of this topic [42]). The related analysis of the Lie-algebraic background of the exact solvability of the one-dimensional Schrödinger equations has been also recently extended to the case of the \mathcal{PT} -symmetric models by Bagchi and Quesne who revealed that a weakening of the Hermiticity requirement implies that the solvable models will form a broader class exhibiting, i.a., an enrichment of their symmetry algebras by complexifying the standard $so(2, 1)$ to $sl(2, C)$ [43].

In the Lie-algebraic setting, a particularly useful role seems to be played by the particular Calogero-type models which mimic a realistic multiparticle dynamics in one dimension. One might note that a suitable \mathcal{PT} -symmetric complexification of these models has been shown also to open a path towards a nonstandard limiting transition to the entirely new solvable models [44].

6.2 \mathcal{PT} -symmetry in combination with SUSY

It is rather amusing to note [45] that the 1998 letter [46] (which might be thought of as one of the very first texts on \mathcal{PT} -symmetry in SUSY systems) paid its attention to only too many features of the problem at once so that, e.g., the exactly solvable model it describes in its last chapter seems to be almost forgotten at present.

One of explanations is that both its second and third authors have already left

physics completely. In contrast, its first author remains extremely active in the field and should be acknowledged for having introduced the author of this lecture in the field in 1999. Unfortunately, once this summary of my Prague’s lecture has a very restricted number of pages, the lack of space forced me to skip virtually all the technical details of my own papers, and the more so in the case of many relevant and extremely interesting results produced, e.g., by F. Cannata and his co-authors [47], by C. Quesne and her co-authors [48] etc.

6.3 Outlook

Let us summarize that when working simply with the two alternative metric operators, one of them may remain indeterminate. For the purely practical purposes it is only necessary that its structure is “sufficiently simple” (typical examples: the Bender’s parity P or the Feshbach–Villars’ σ_3). The second metric should then be constructed as positively definite, allowing us to define the norms of states. One can hardly expect that the latter operator would be not too complicated.

What marks the progress in the whole theory of this type is the presentation of several explicit examples of the desirable physical metrics. Besides the early product QP (assigned to the exactly solvable \mathcal{PT} -symmetric spiked harmonic oscillator in 1999 [22]), one should not forget the Ali Mostafazadeh’s alternative to the Feshbach–Villars’ metric for the Klein–Gordon field [40] and, last but not least, several fresh, beautiful and explicit constructions of the products CP obtained by different sophisticated methods for several different field models by Carl Bender and his co-authors [49].

On this background, the standard studies of SUSY models are also re-acquiring a new motivation. The present review mentioned the few steps in this direction, exhibiting already certain clear parallels between pseudo- and Hermitian SUSY. By the present author’s opinion, these preliminary sample results just mark a very start of a more intensive development of this subject in some very near future.

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