The second term of the semi-classical asymptotic expansion of Feynman path integrals with integrand of polynomial growth.

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Recently N. Kumano-go succeeded in proving that piecewise linear time slicing approximation to Feynman path integral with integrand $F(\gamma)$ actually converges to the limit as the mesh of division of time goes to 0 if the functional $F(\gamma)$ of paths γ belongs to a certain class of functionals with polynomial growth at the infinity. Moreover, he rigorously showed that the limit, which we call the Feynman path integral, has rich properties.

The aim of this note is to explain that the use of piecewise classical paths naturally leads us to an analytic formula for the second term of the semi-classical asymptotic expansion of the Feynman path integrals under a little stronger assumptions than that in Kumano-go's. If $F(\gamma) \equiv 1$, this second term coincides with the one given by G.D. Birkhoff.

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1 Introduction

A path γ is a continuous or sufficiently smooth map from the time interval $[s, s']$ to the configuration space $\mathbf{R}^d$. The action $S(\gamma)$ of a path γ is the integral

$$S(\gamma) = \int_s^{s'} L\left(t, \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t), \gamma(t)\right)\mathrm{d}t \tag{1.1}$$

along γ of the Lagrangian

$$L(t, \dot{x}, x) = \tfrac{1}{2}\,|\dot{x}|^2 - V(t, x)\,,$$

with the potential $V(t, x)$. We assume $V(t, x)$ is a function continuous with respect to the variables $(t, x) \in \mathbf{R} \times \mathbf{R}^d$ and infinitely differentiable with respect to $x \in \mathbf{R}^d$.

In this note we discuss Feynman path integral

$$\int_\Omega \mathrm{e}^{\mathrm{i}\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma]\,, \tag{1.2}$$

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where $\nu = 2\pi/h$ with Planck constant h.

We assume in this note that the potential satisfies the following assumption: For any non-negative integer m there exists a non-negative constant v_m such that

$$\max_{|\alpha|=m}\Bigl(\sup_{(t,x)\in[0,T]\times\mathbf{R}^d}|\partial_x^\alpha V(t,x)|\Bigr) \le v_m(1+|x|)^{\max\{2-m,0\}}\,. \tag{1.3}$$

This assumption is close to that of Pauli in [3]. Let $[s,s']$ be an interval of time. A path γ is classical if it is a solution to the Euler equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\gamma(t) + (\nabla V)\bigl(t,\gamma(t)\bigr) = 0 \quad \text{for } s < t < s'\,. \tag{1.4}$$

Here and hereafter ∇ stands for the nabla operator in the configuration space $\mathbf{R}^d$. For arbitrary pair of points $x, y \in \mathbf{R}^d$ there exists one and only one classical path γ that satisfies the boundary condition

$$\gamma(s) = y\,, \quad \gamma(s') = x\,, \tag{1.5}$$

if $|s'-s| \le \mu$ with sufficiently small μ, say for instance

$$\mu^2 d v_2 < 1\,. \tag{1.6}$$

In this case the action $S(\gamma)$ of γ is a function of (s', s, x, y) and is denoted by $S(s', s, x, y)$, i.e.,

$$S(s',s,x,y) = \int_s^{s'} L\biggl(t, \frac{\mathrm{d}}{\mathrm{d}t}\,\gamma(t), \gamma(t)\biggr)\mathrm{d}t\,. \tag{1.7}$$

Since it was shown by [2] (see also [14]) that Feynman's path integral is not a measure theoretic integral, we must give meaning to the integral (1.2). Among several ways to give meaning to the Feynman path integrals (1.2) we adopt here the time slicing approximation method, which Feynman himself used in [4]. We recall this method. Let

$$\varDelta : 0 = T_0 < T_1 < \cdots < T_J < T_{J+1} = T \tag{1.8}$$

be a division of the interval $[0,T]$. Then we set $t_j = T_j - T_{j-1}$ and define the mesh $|\varDelta|$ of the division $\varDelta$ by $|\varDelta| = \max_j\{t_j\}$. We always assume that

$$|\varDelta| \le \mu\,. \tag{1.9}$$

Let

$$x_j \in \mathbf{R}^d\,, \quad j = 0, 1, \ldots, J, J+1\,, \tag{1.10}$$

be arbitrary $J+2$ points of the configuration space $\mathbf{R}^d$. The piecewise classical path $\gamma_\varDelta$ with vertices $(x_{J+1}, x_J, \ldots, x_1, x_0) \in \mathbf{R}^{d(J+2)}$ is the broken path that satisfies the Euler equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\gamma_\varDelta(t) + (\nabla V)\bigl(t,\gamma_\varDelta(t)\bigr) = 0\,, \tag{1.11}$$

for $T_{j-1} < t < T_j$, $j = 1, 2, \ldots, J, J+1$, and boundary conditions

$$\gamma_\Delta(T_j) = x_j\,, \quad j = 0, 1, \ldots, J, J+1\,, \tag{1.12}$$

where $x = x_{J+1}$ and $y = x_0$. When we wish to emphasize the fact that this path γ_Δ depends on $(x_{J+1}, x_J, \ldots, x_1, x_0)$, we denote it by $\gamma_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ or $\gamma_\Delta(t; x_{J+1}, x_J, \ldots, x_1, x_0)$, where t is the time variable.

Let $F(\gamma)$ be a functional defined for paths γ. Its value $F(\gamma_\Delta)$ at γ_Δ can be written as a function $F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ of $(x_{J+1}, x_J, \ldots, x_1, x_0)$. For example the action functional $S(\gamma_\Delta)$ of γ_Δ is given by

$$\begin{aligned} S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0) = S(\gamma_\Delta) &= \int_0^T L\left(t, \frac{\mathrm{d}}{\mathrm{d}t}\gamma_\Delta(t), \gamma_\Delta(t)\right)\mathrm{d}t = \\ &= \sum_{j=1}^{J+1} S_j(x_j, x_{j-1})\,, \end{aligned} \tag{1.13}$$

where we used the abbreviation

$$S_j(x_j, x_{j-1}) = S(T_j, T_{j-1}, x_j, x_{j-1}) = \int_{T_{j-1}}^{T_j} L\left(t, \frac{\mathrm{d}}{\mathrm{d}t}\gamma_\Delta(t), \gamma_\Delta(t)\right)\mathrm{d}t\,. \tag{1.14}$$

A piecewise classical time slicing approximation to Feynman path integral (1.2) with the integrand $F(\gamma)$ is an oscillatory integral

$$\begin{aligned} I[F_\Delta](\Delta; x, y) &= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi \mathrm{i} t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}} \mathrm{e}^{\mathrm{i}\nu S(\gamma_\Delta)} F(\gamma_\Delta)\prod_{j=1}^{J}\mathrm{d}x_j = \\ &= \prod_{j=1}^{J+1}\left(\frac{\nu}{2\pi \mathrm{i} t_j}\right)^{d/2}\int_{\mathbf{R}^{dJ}} \mathrm{e}^{\mathrm{i}\nu S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)} F_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)\prod_{j=1}^{J}\mathrm{d}x_j\,, \end{aligned} \tag{1.15}$$

where $x_{J+1} = x$ and $x_0 = y$. See Feynman [4].

Feynman's definition of path integral (1.2) is

$$\int_\Omega \mathrm{e}^{\mathrm{i}\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma] = \lim_{|\Delta| \to 0} I[F_\Delta](\Delta; x, y)\,, \tag{1.16}$$

if the limit on the right hand side exists.

We remark that Feynman [4] used also piecewise linear paths in place of piecewise classical paths. In that case we say piecewise linear time slicing approximation method.

Existence of the limit in (1.16) was proved in the case $F \equiv 1$ by [5–7, 17]. Recently N.Kumano-go [16] proved the limit in (1.16) exists in the case of more general class of functional F using piecewise linear paths in place of piecewise classical paths.

2 Results

We assume that the potential satisfies the assumption (1.3) and μ satisfies (1.6). Let $\varDelta$ be as (1.8) and (1.9). Then the set $\Gamma(\varDelta)$ of all piecewise classical paths associated with the division $\varDelta$ forms a differentiable manifold of dimension $d(J+2)$. The correspondence $\gamma_\varDelta \to (x_{J+1}, \ldots, x_0)$ is a global coordinate system. We will describe a basis of the tangent space $T_{\gamma_\varDelta}\Gamma(\varDelta)$ to $\Gamma(\varDelta)$ at $\gamma_\varDelta$. Let $\{e_k\}_{k=1}^d$ be an orthonormal frame of the configuration space $\mathbf{R}^d$, i.e., $x_j = \sum_{k=1}^d x_{j,k}e_k$ in our notation (1.10). Let $\eta_{j;k}(t) = \partial_{x_{j,k}}\gamma_\varDelta(t)$. Then the functions $\{\eta_{j;k}\}_{0\le j\le J+1\ \ 1\le k\le d}$ form a basis of the tangent space $T_{\gamma_\varDelta}\Gamma(\varDelta)$. If $j = 1, \ldots, J$, then for $t \le T_{j-1}$ or $T_{j+1} \le t$,

$$\eta_{j;k}(t) = 0\,, \tag{2.1}$$

and for $T_{j-1} < t < T_j$ or $T_j < t < T_{j+1}$ it satisfies Jacobi equation at $\gamma_\varDelta$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\eta_{j;k}(t) + \nabla\nabla V\big(t, \gamma_\varDelta(t)\big)\eta_{j;k}(t) = 0\,, \tag{2.2}$$

and at $t = T_j$ it satisfies the boundary condition:

$$\eta_{j;k}(T_j) = e_k\,. \tag{2.3}$$

If $j = 0$, $\eta_{0;k}(t) = 0$ for $T_1 \le t$, (2.2) is satisfied for $0 < t < T_1$ and (2.3) is satisfied at $t = T_0$. If $j = J+1$, $\eta_{J+1;k}(t) = 0$ for $t \le T_J$, (2.2) is satisfied for $T_J < t < T_{J+1}$ and (2.3) is satisfied at $t = T_{J+1}$.

For a pair of divisions $\varDelta'$ and $\varDelta$ we use symbol $\varDelta \prec \varDelta'$ if $\varDelta'$ is a refinement of $\varDelta$. If $\varDelta \prec \varDelta'$, then there is a natural inclusion $\Gamma(\varDelta) \subset \Gamma(\varDelta')$. This inclusion induces inclusion relation of the tangent spaces at $\gamma_\varDelta$, i.e., $T_{\gamma_\varDelta}\Gamma(\varDelta) \subset T_{\gamma_\varDelta}\Gamma(\varDelta')$. The set Γ of all piecewise classical paths is the inductive limit of $\{\Gamma(\varDelta), \prec\}$, i.e., $\Gamma = \varinjlim \Gamma(\varDelta)$. Γ is a dense subset of the Sobolev space $H^1([0,T]; \mathbf{R}^d)$ of order 1 with values in $\mathbf{R}^d$ and hence it is also dense in the space $C([0,T]; \mathbf{R}^d)$ of all continuous paths. Let $\gamma_\varDelta \in \Gamma(\varDelta)$. Then the tangent space $T_{\gamma_\varDelta}\Gamma$ to Γ at $\gamma_\varDelta$ is the inductive limit $\varinjlim T_{\gamma_\varDelta}\Gamma(\varDelta)$, which is a dense linear subspace of the Sobolev space $H^1([0,T]; \mathbf{R}^d)$.

Let $F(\gamma)$ be a functional defined on Γ. We denote its differential at $\gamma \in \Gamma$ by DF_γ if it exists. And $DF_\gamma[\zeta]$ stands for its value at the tangent vector $\zeta \in T_\gamma\Gamma$. For any integer $n > 0$ and for $\zeta_j \in T_\gamma\Gamma$, $j = 1, 2, \ldots, n$, we denote by $D^nF_\gamma[\zeta_1 \otimes \zeta_2 \otimes \ldots \otimes \zeta_n]$, the symmetric n-linear form on the tangent space arising from the n-th jet modulo $(n-1)$-th jet of F at γ.

We assume always in this paper that the functional $F(\gamma)$ satisfies both of the following conditions.

Assumption 1 *Let $m \ge 0$. For any non-negative integer K there exist positive constants A_K and X_K such that for any division $\varDelta$ of the form (1.8) and any* *integer* n_j, $0 \le j \le J+1$, *with* $0 \le n_j \le K$

$$\begin{aligned} &\Big| D^{\Sigma_{j=0}^{J+1} n_j} F_{\gamma_\Delta}\left[\otimes_{j=0}^{J+1} \otimes_{k=1}^{n_j} \zeta_{j,k}\right] \Big| \le \\ &\qquad \le A_K X_K^{J+2}\big(1+\|\gamma_\Delta\| + |||\gamma_\Delta|||\big)^m \prod_{j=0}^{J+1}\prod_{k=1}^{n_j} \|\zeta_{j,k}\| , \end{aligned} \tag{2.4}$$

as far as $\zeta_{j,k} \in T_{\gamma_\Delta}\Gamma$ *satisfies*

$$\operatorname{supp}\zeta_{j,k} \subset \begin{cases} [0,T_1]\,, & \text{if} \quad j=0\,, \\ [T_{j-1},T_{j+1}]\,, & \text{if} \quad 1\le j\le J\,, \\ [T_J,T_{J+1}]\,, & \text{if} \quad j=J+1\,, \end{cases} \tag{2.5}$$

where $\|\zeta\| = \max_{0\le t\le T} |\zeta(t)|$ *and* $|||\gamma_\Delta||| =$ *total variation of* γ_Δ.

Assumption 2 *[10,16]. There exists a positive bounded Borel measure* ρ *on* $[0,T]$ *such that with the same* A_K, X_K *as above*

$$\begin{aligned} &\Big| D^{1+\Sigma_{j=0}^{J+1} n_j} F_{\gamma_\Delta}\left[\eta\otimes\otimes_{j=0}^{J+1} \otimes_{k=0}^{n_j} \zeta_{j,k}\right] \Big| \le \\ &\quad \le A_K X_K^{J+2}\big(1+\|\gamma_\Delta\| + |||\gamma_\Delta|||\big)^m \int_{[0,T]} |\eta(t)|\rho(\mathrm{d}t) \prod_{j=0}^{J+1}\prod_{k=0}^{n_j} \|\zeta_{j,k}\| \end{aligned} \tag{2.6}$$

for any division Δ, *integer* $n_j \le K$ *and* $\zeta_{j,k}$ *which are the same as in Assumption 1. And* η *is also an arbitrary element of* $T_{\gamma_\Delta}\Gamma$.

We can prove the following

Theorem 1 *Assume that the integrand* $F(\gamma)$ *satisfies Assumption 1 and Assumption 2 above and* T *is so small that* $|T| \le \mu$. *Then the limit of the right hand side of (1.16) converges compact-uniformly with respect to* $(x,y) \in \mathbf{R}^{2d}$.

We shall make more precise statement. We fix (x,y). We assume that $|T| \le \mu$. Then the action $S(\gamma)$ has the unique critical point γ^*, which is the unique classical path starting y at time 0 and reaching x at time T. The critical point is non-degenerate. Similarly, if $T \le \mu$ the function $S_\Delta(x_{J+1}, x_J, \ldots, x_1, x_0)$ of $(x_J, \ldots, x_1)$ has only one critical point, which is non-degenerate. So we can apply stationary phase method to (1.15) and obtain the following expression:

$$I[F_\Delta](\Delta;x,y) = \left(\frac{\nu}{2\pi \mathrm{i} T}\right)^{d/2} D(\Delta;x,y)^{-1/2} \mathrm{e}^{\mathrm{i}\nu S(\gamma^*)}\Big(F(\gamma^*) + \nu^{-1} R_\Delta[F_\Delta](\nu,x,y)\Big). \tag{2.7}$$

Here we used the following symbol

$$D(\Delta;x,y) = \left(\frac{t_{J+1}t_J\ldots t_1}{T}\right)^d \det\operatorname{Hess} S(\gamma_\Delta)\,, \tag{2.8}$$

where Hess $S(\gamma_\Delta)$ denotes the Hessian of $S(\gamma_\Delta)$ with respect to $(x_J, x_{J-1}, \dots, x_1)$. It is shown in [8] and [16] that for any non-negative integer K there exist a positive constant C_K and a positive integer $M(K)$ independent of ν and of Δ such that

$$\left|\,\partial_x^\alpha \partial_y^\beta R_\Delta[F_\Delta](\nu, x, y)\right| \le C_K A_{M(K)} T\big(T + \rho([0,T])\big) bigl(1 + |x| + |y|\big)^m , \quad (2.9)$$

as far as $|\alpha|, |\beta| \le K$. The function $D(\Delta; x, y)$ is of the form (cf. [7])

$$D(\Delta; x, y) = 1 + T^2 d(\Delta; x, y) . \quad (2.10)$$

For any multi-indices α, β there exists a positive constant $C_{\alpha,\beta}$ such that

$$\left|\,\partial_x^\alpha \partial_y^\beta d(\Delta; x, y)\right| \le C_{\alpha,\beta} . \quad (2.11)$$

We also know (cf. [7]) that $D(T, x, y) = \lim_{|\Delta| \to 0} D(\Delta; x, y)$ exists. Moreover, for any multi-indices α, β there exits a non-negative constant $C_{\alpha,\beta}$ such that

$$\left|\,\partial_x^\alpha \partial_y^\beta \left(D(T, x, y) - D(\Delta; x, y)\right)\right| \le C_{\alpha,\beta} |\Delta| T . \quad (2.12)$$

$D(T, x, y)$ is of the form

$$D(T, x, y) = 1 + T^2 d(T, x, y) , \quad (2.13)$$

where $d(T, x, y)$ satisfies the same estimate as (2.11).

The estimate (2.12) was proved earlier in [7]. The function $T^{-n} D(T, x, y)$ is the Morette–VanVleck determinant (cf. [7]).

Theorem 2 *Under the Assumptions 1 and 2 we can write the limit* $\lim\limits_{|\Delta| \to 0} I[F_\Delta](\Delta; x, y)$ *in the following way:*

$$\begin{aligned} \int_\Omega \mathrm{e}^{\mathrm{i}\nu S(\gamma)} F(\gamma) \mathcal{D}[\gamma] &= \lim_{|\Delta| \to 0} I[F_\Delta](\Delta; x, y) = \\ &= \left(\frac{\nu}{2\pi \mathrm{i} T}\right)^{d/2} D(T, x, y)^{-1/2} \mathrm{e}^{\mathrm{i}\nu S(\gamma^*)} \Big(F(\gamma^*) + \nu^{-1} R[F](\nu, x, y)\Big) . \end{aligned} \quad (2.14)$$

Moreover, for any non-negative integer K there exist positive constant C_K and a non-negative integer $M(K)$ independent of ν and of Δ such that

$$\begin{aligned} &\left|\,\partial_x^\alpha \partial_y^\beta \left(R[F](\nu, x, y) - R_\Delta[F_\Delta](\nu, x, y)\right)\right| \le \\ &\le C_K A_{M(K)} |\Delta| \left(\rho([0,T]) + T^2 + T^3 + T^2 \rho([0,T]) + T\nu^{-1}\right) \left(1 + |x| + |y|\right)^m . \end{aligned} \quad (2.15)$$

as far as $|\alpha| \le K$ and $|\beta| \le K$.

Corollary 1 *For any non-negative integer K there exist a positive constant C_K and a non-negative integer $M(K)$ independent of ν such that*

$$\left|\,\partial_x^\alpha \partial_y^\beta R[F](\nu, x, y)\right| \le C_K A_{M(K)} T\big(T + \rho([0,T])\big)\big(1 + |x| + |y|\big)^m . \quad (2.16)$$

It is expected that the following semi-classical asymptotic expansion holds;

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma] = \tag{2.17}$$
$$= \Big(\frac{\nu}{2\pi i T}\Big)^{d/2} D(T,x,y)^{-1/2} e^{i\nu S(\gamma^*)}\Big(A_0 + \nu^{-1}A_1 + O(\nu^{-2})\Big),$$

as $\nu \to \infty$.

Theorem 2 implies $A_0 = F(\gamma^*)$. What is the next term A_1?

In the case $F(\gamma) \equiv 1$ assuming the existence of expansion, Birkhoff gave answer [1]. In fact, he gave even higher order terms of asymptotic expansion. However, if $F(\gamma) \neq$ constant, then his method is not available.

We write down the second term A_1 of (2.17) for general $F(\gamma)$ and prove that the asymptotic expression is actually holds. For this purpose we shall make a preparation of notations. Let ϵ be an arbitrary small positive number. And $\Delta(t,\epsilon)$ be the division

$$\Delta(t,\epsilon) : 0 = T_0 < t < t+\epsilon < T\,. \tag{2.18}$$

Let z be an arbitrary point in $\mathbf{R}^d$. We abbreviate the piecewise classical path $\gamma_{\Delta(t,\epsilon)}\big(s,x,\gamma^*(t+\epsilon),z,y\big)$ associated with the division $\Delta(t,\epsilon)$ by $\gamma_{\{t,\epsilon\}}(s,z)$, i.e., $\gamma_{\{t,\epsilon\}}(s,z)$ is the piecewise classical path which satisfies conditions:

$$\gamma_{\{t,\epsilon\}}(0,z) = y\,,\quad \gamma_{\{t,\epsilon\}}(t,z) = z\,,\quad \gamma_{\{t,\epsilon\}}(t+\epsilon,z) = \gamma^*(t+\epsilon)\,,\quad \gamma_{\{t,\epsilon\}}(T,z) = x\,. \tag{2.19}$$

It is clear that $\gamma_{\{t,\epsilon\}}(s,z)$ coincides with $\gamma^*(s)$ for $t+\epsilon \le s \le T$ independent of z. Therefore, $\partial_z\gamma_{\{t,\epsilon\}}(s,z) = 0$ for $t+\epsilon \le s \le T$.

Lemma 1 *Under the Assumptions 1 and 2 there exists the limit*

$$q(t) = \lim_{\epsilon\to+0}\Big[\Delta_z\Big(D(t,z,y)^{-1/2}F\big(\gamma_{\{t,\epsilon\}}(*,z)\big)\Big)\Big|_{z=\gamma^*(t)}\Big], \tag{2.20}$$

where Δ_z stands for the Laplacian with respect to z.

Theorem 3 *In addition to our Assumptions 1 and 2 we further assume that the function $q(t)$ of Lemma 1 is Riemannian integrable over $[0,T]$. Set*

$$A_1 = \frac{i}{2}\int_0^T D\big(t,\gamma^*(t),y\big)^{1/2} q(t)\,dt\,. \tag{2.21}$$

Then, there holds the asymptotic formula, as $\nu \to \infty$,

$$\int_\Omega e^{i\nu S(\gamma)} F(\gamma)\mathcal{D}[\gamma] = \tag{2.22}$$
$$= \Big(\frac{\nu}{2\pi i T}\Big)^{d/2} D(T,x,y)^{-1/2} e^{i\nu S(\gamma^*)}\Big(A_0 + \nu^{-1}A_1 + \nu^{-2}r(\nu,x,y)\Big),$$

where for any α, β the remainder term $r(\nu,x,y)$ satisfies estimate

$$\big|\,\partial^\alpha\partial^\beta r(\nu,x,y)\big| \le C_{\alpha,\beta}T^2\big(1+|x|+|y|\big)^m\,. \tag{2.23}$$

Although our method is completely different from Birkhoff's method, our formula coincides with Birkhoff's result in the case of $F(\gamma) \equiv 1$.

More detailed discussions are given in our paper [12], which heavily uses the result of [11].

Remark 1 *In this note the Lagrangian has no vector potential. Kitada–Kumano-go [15], Yajima [18] and Tshuchida–Fujiwara [13] discussed the case of Lagrangian with non zero vector potential. They proved that the limit (1.16) exists and the limit is the fundamental solution of Schrödinger equation if* $F(\gamma) \equiv 1$. *However we do not know whether the limit (1.16) exists or not if* $F(\gamma) \neq \text{constant}$ *and Lagrangian has non-zero vector potential.*

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